Generalizing Inflation:

Constraining Correlations in General Causal Structures for Various Physical Theories

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Collaborators (GPT results)

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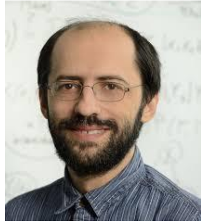
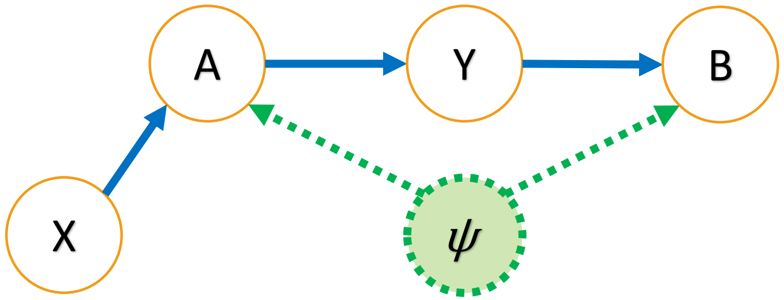
Vienna, Austria 	Brussels, Belgium 	Baltimore, MD USA 

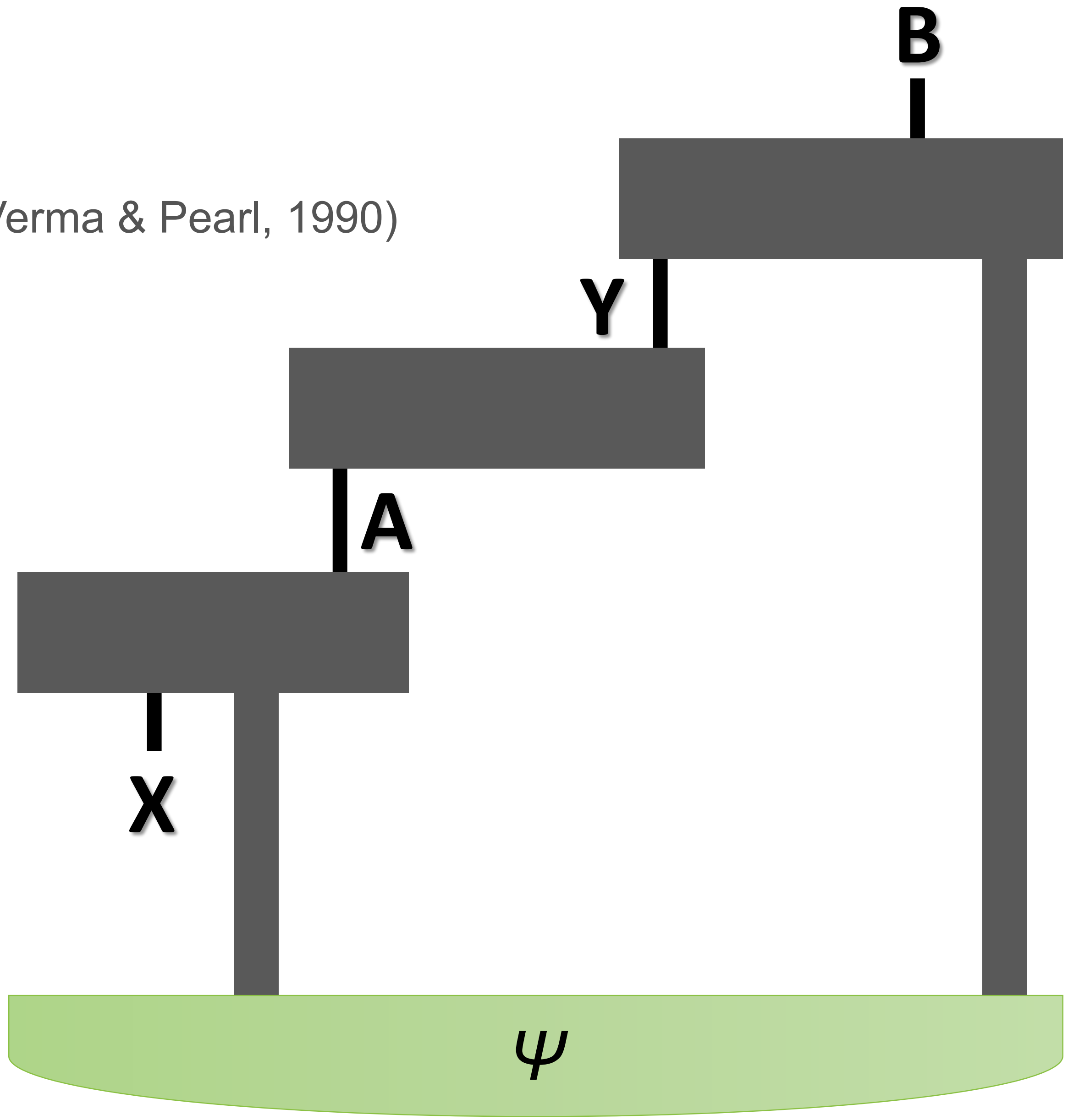
Part One

Quantum Instrumental Inequalities, etc.

Verma Scenario

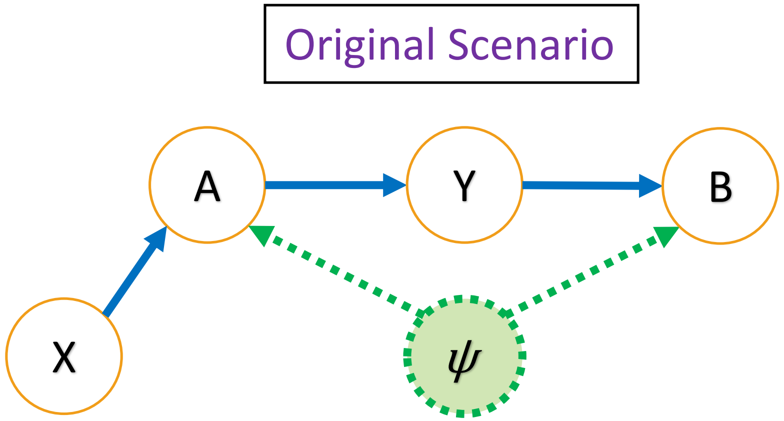
"Equivalence and Synthesis of Causal Models" (Verma & Pearl, 1990)

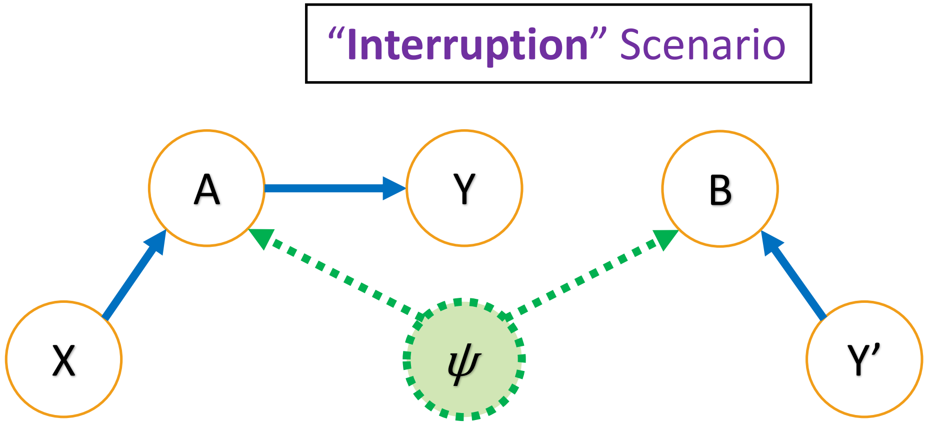




Seems like should be able to isolate the nonclassical (Bell-like) part through knowing how $\boldsymbol{Y}$ classically depends on $\boldsymbol{A}$...

Verma Scenario



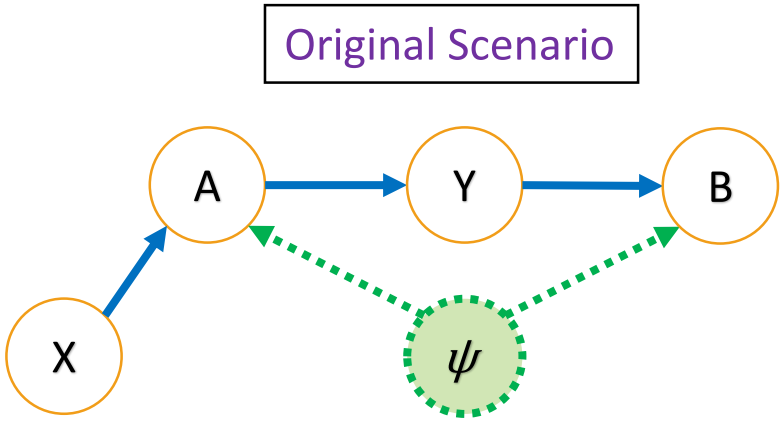


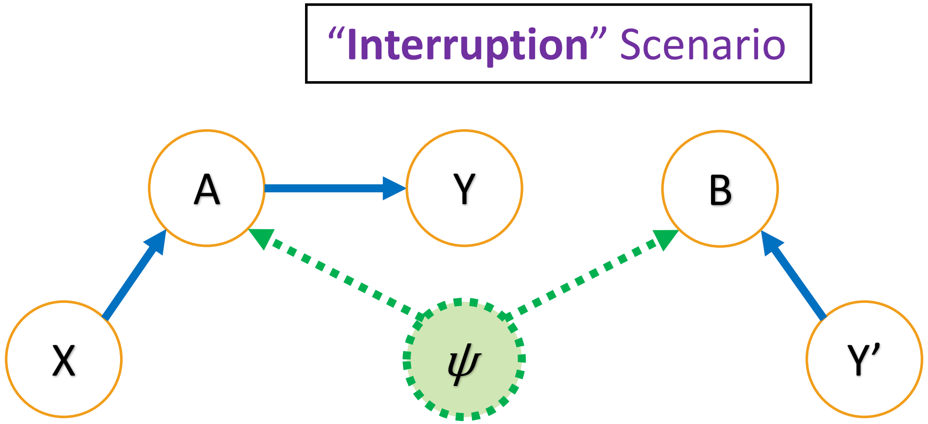
$P_{Orig}(A{=}a, B{=}b, Y{=}y | X{=}x) \sim G_{Original}$ iff

$\exists \; P_{Interruption}(A{=}a, B{=}b, Y{=}y | X{=}x, Y'{=}y') \sim G_{Interruption}$

$P_{Orig}(A{=}a, B{=}b, Y{=}y | X{=}x) = P_{Interruption}(A{=}a, B{=}b, Y{=}y | X{=}x, Y'{=}y)$

Verma Scenario





$P_{Orig}(A{=}a, B{=}b, Y{=}y|X{=}x) \sim G_{Original}$ iff

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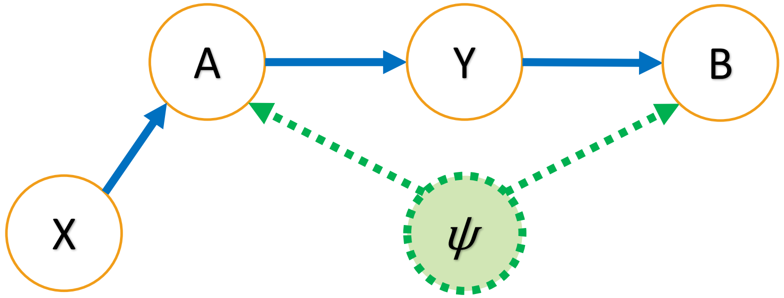
$P_{Orig}(A{=}a, B{=}b, Y{=}y|X{=}x) = P_{Interruption}(A{=}a, B{=}b, Y{=}y|X{=}x, Y'{=}y)$

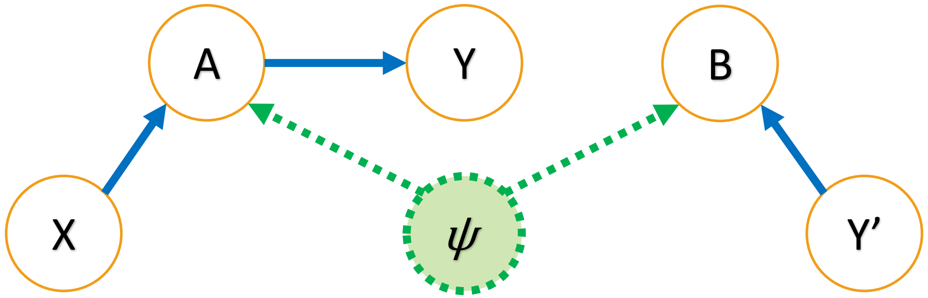
$P_{Interruption}(A, B, Y|X, Y')$ is **unique**, and identified as

$$P_{Interruption}(A, B, Y|X, Y') \equiv \frac{P_{Orig}(A, B, Y{=}y'|X)}{P_{Orig}(Y{=}y'|A)} P_{Orig}(Y{=}y|A)$$

Verma Scenario

What??





We are exploiting **truncated factorization** to isolate the elementary **functional constituents** of a causal model. Basically,

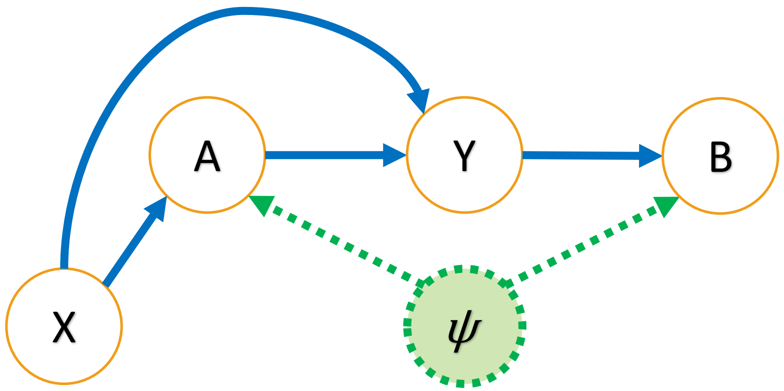
$$P_{Orig}(A,B,X,Y) = P_{Orig}(X)P_{Orig}\big(Y|do(A)\big)P_{Orig}\big(A,B|do(XY)\big) \quad \text{where } P_{Orig}\big(Y|do(A)\big) = P_{Orig}(Y|A)$$

&

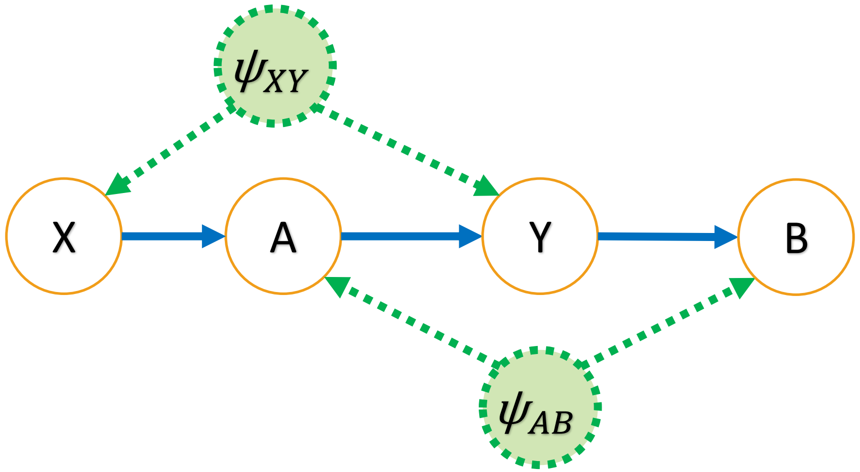
$$P_{Interruption}(A,B,X,Y,Y') = P_{Orig}(X)P_{Orig}\big(Y|do(A)\big)P_{Orig}\big(A,B|do(XY')\big)$$

See "Identification of Conditional Interventional Distributions" (Shpitser & Pearl 2006)
"Introduction to Nested Markov Models" (Shpitser et. al. 2014)

Other Verma Scenarios



$$P_{Orig}(AB|do(XY)) = \frac{P_{Orig}(ABY|X)}{P_{Orig}(Y|AB)}$$

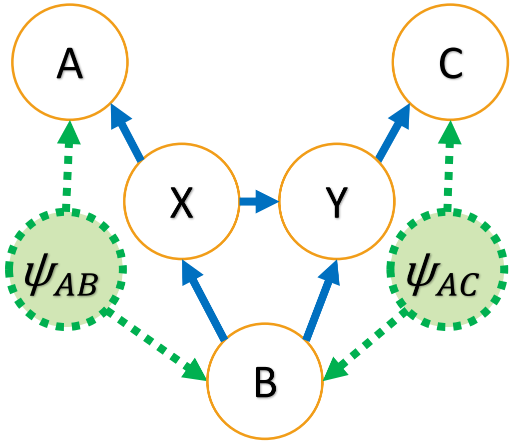


$$P_{Orig}(AB|do(XY)) = \frac{P_{Orig}(ABXY)P_{Orig}(A|X)}{P_{Orig}(AXY)}$$

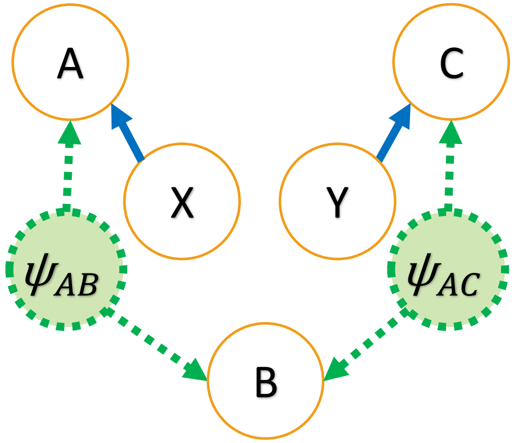
In both these cases $P_{Orig}(AB|do(XY))$ is **identifiable,** and must be compatible with quantum Bell scenario.

Bilocality Kernel

Original Scenario



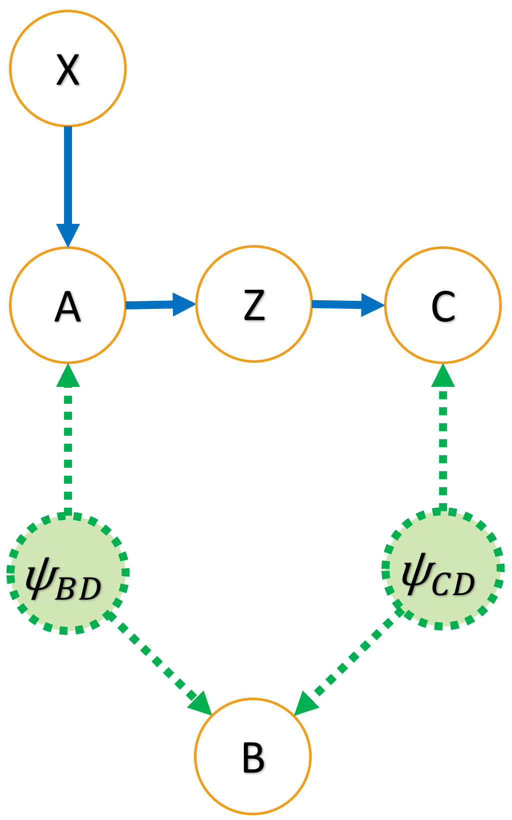
"**Interruption**" Scenario



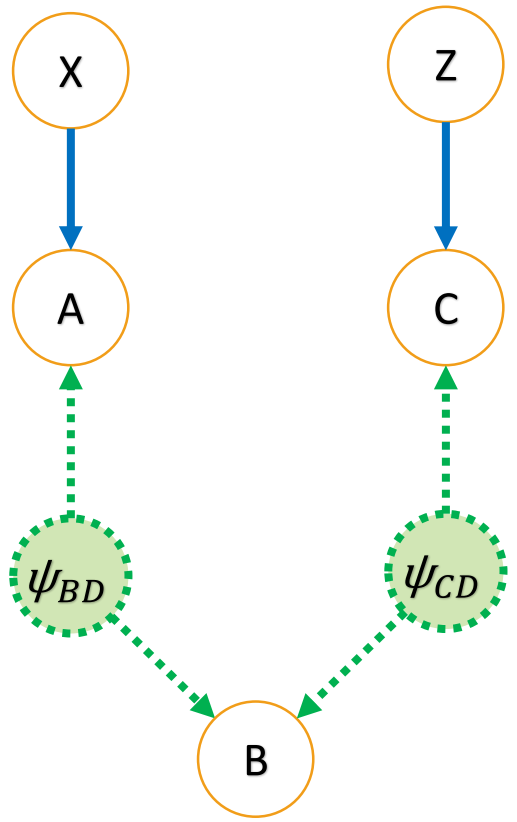
$$P_{Orig}(ABC|do(XY)) = \frac{P_{Orig}(ABCXY)}{P_{Orig}(X|B)P_{Orig}(Y|XB)}$$

Another Bilocality Kernel

Original Scenario

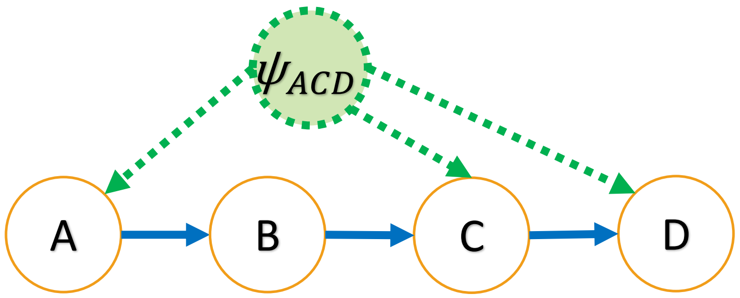


“**Interruption**” Scenario

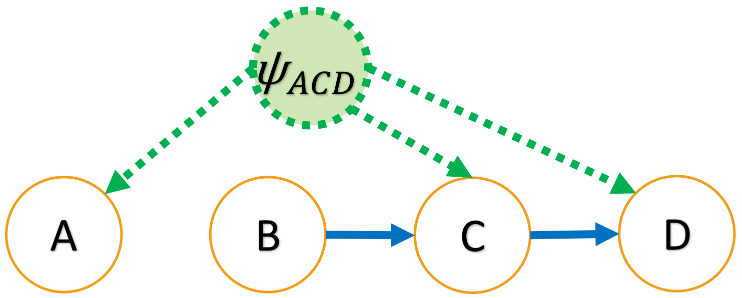


…and another…

Original Scenario

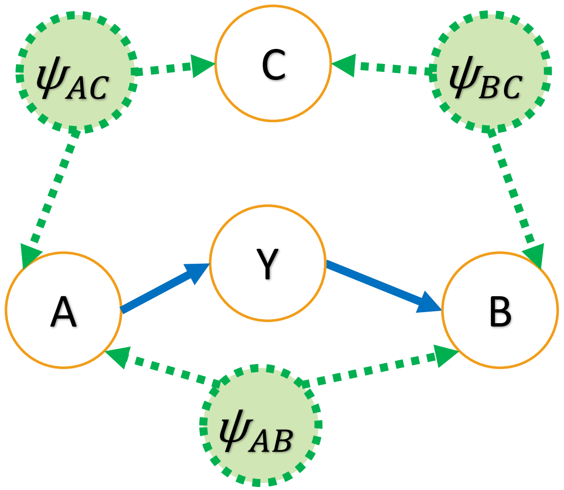


“**Interruption**” Scenario

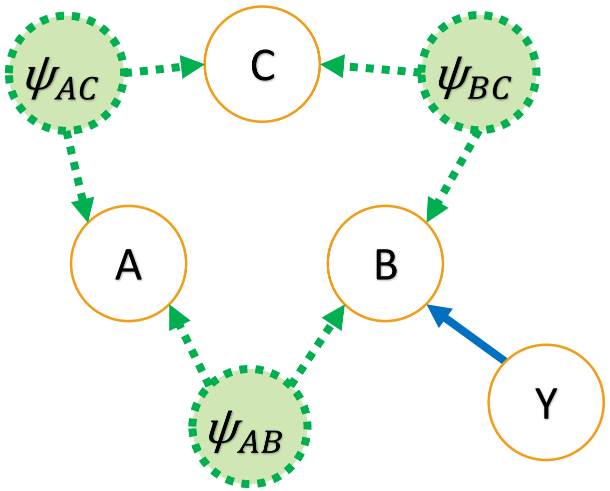


Triangle Kernel

Original Scenario



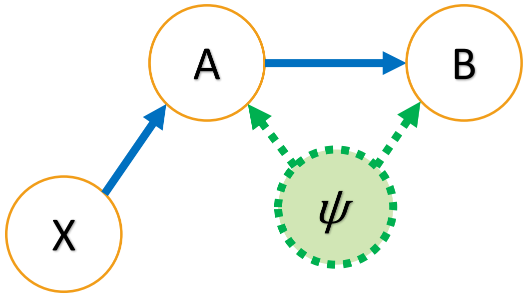
"**Interruption**" Scenario



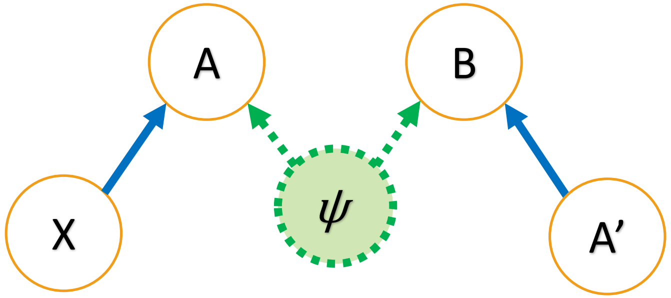
$$P_{Orig}(ABC|do(Y)) = \frac{P_{Orig}(ABCY)}{P_{Orig}(Y|B)}$$

Instrumental Scenario

Original Scenario



"**Interruption**" Scenario



Here, $P_{Orig}(B|do(A))$ is not identified, but it is **constrained** via

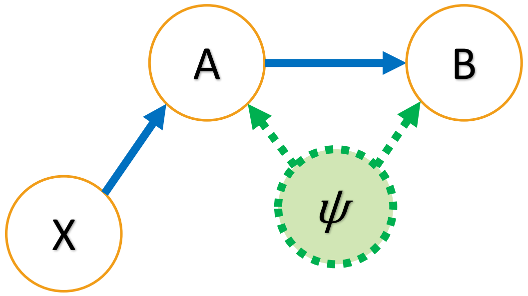
$$\exists \ P_{Interruption}(A{=}a, B{=}b|X{=}x, A'{=}a') \sim G_{Interruption}$$

such that

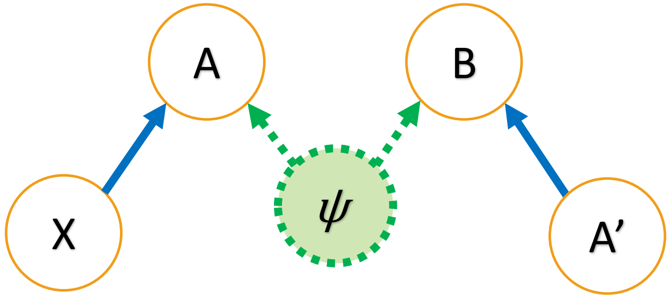
1. $P_{Orig}(A{=}a, B{=}b|X{=}x) = P_{Interruption}(A{=}a, B{=}b|X{=}x, A'{=}a)$
2. $P_{Orig}(B|do(A{=}\text{a})) = P_{Interruption}(B|A'{=}a)$

Instrumental Scenario

Original Scenario



"**Interruption**" Scenario



Thus one can **constrain** classical, quantum, or GPT do-conditionals. **Different** effect bounds!

Do-conditional effect bounds can be converted into compatibility inequalities, e.g.

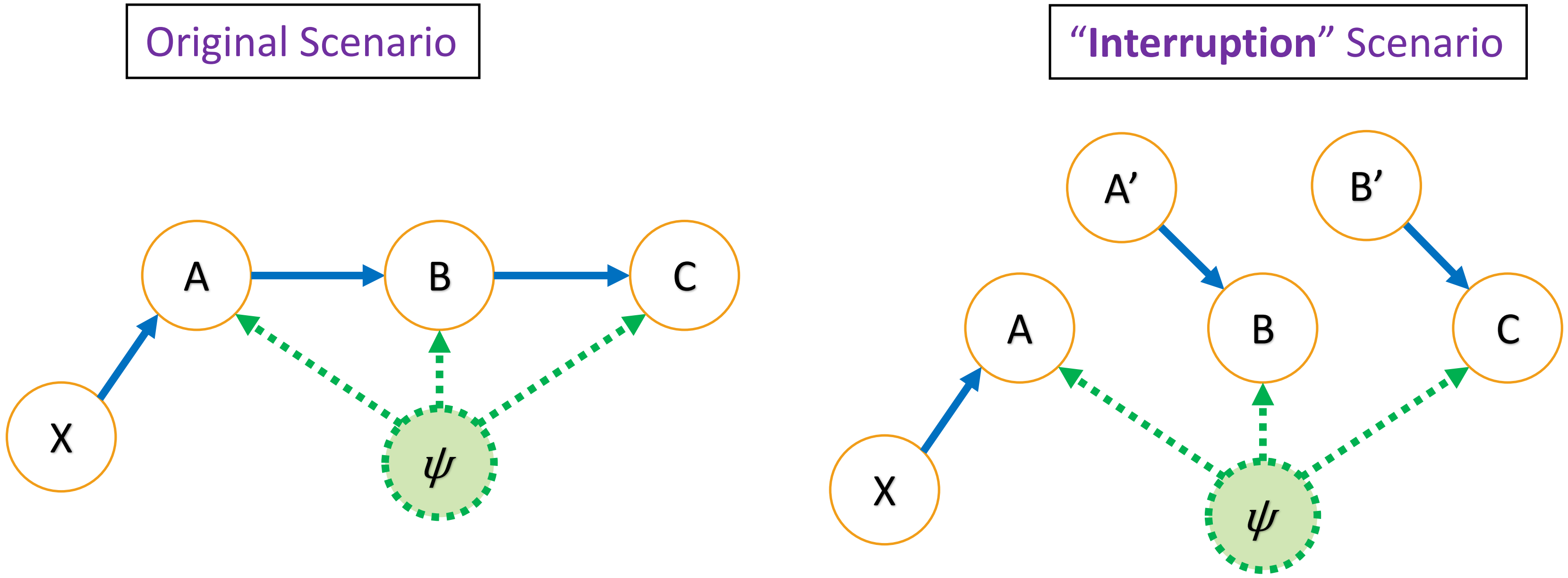
$$\text{LowerBound}\left(P_{Orig}\big(B{=}b|do(A{=}a)\big)\right) \leq \text{UpperBound}\left(P_{Orig}\big(B{=}b|do(A{=}a)\big)\right)$$

(Note that GPT compatibility **equalities** in the interruption scenario translate to **inequalities** in the original scenario.)

References:

- Quantum instrumental effect estimation:
  "Quantum violation of an instrumental test" (Chaves et. al. 2017)
- Relating Instrumental to Bell Scenario:
  "Quantum violations in the Instrumental scenario and their relations to the Bell scenario" (Van Himbeeck et. al. 2018)
- Compatibility inequalities from do-conditional bounds:
  "Bounds on treatment effects from studies with imperfect compliance" (Balke & Pearl, 1997)
- Do-conditional constraints from No-Signalling alone:
  "Inequality Constraints in Causal Models with Hidden Variables" (Kang & Tian, 2006)

Multipartite Bell for constraining do-conditionals



Here, $P_{Orig}(B|do(A))$ and $P_{Orig}(C|do(B))$ are not identified, but are **constrained** via

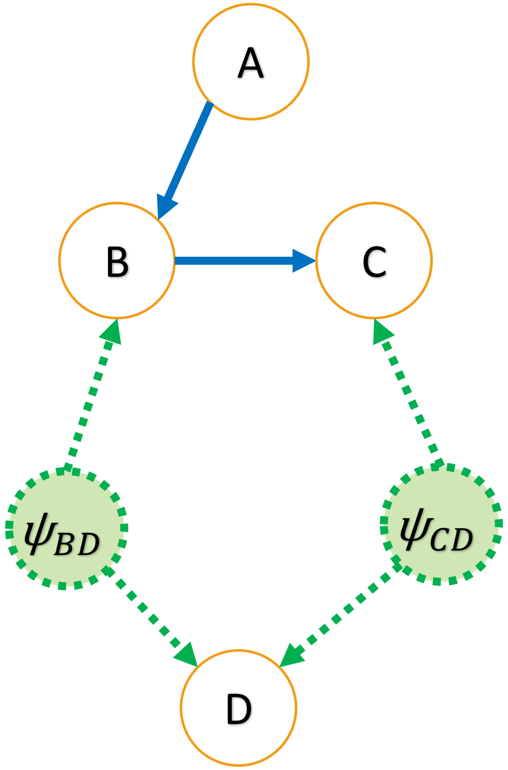
$$\exists \; P_{Interruption}(A{=}a, B{=}b, C{=}c|X{=}x, A'{=}a', B'{=}b') \sim G_{Interruption}$$

such that

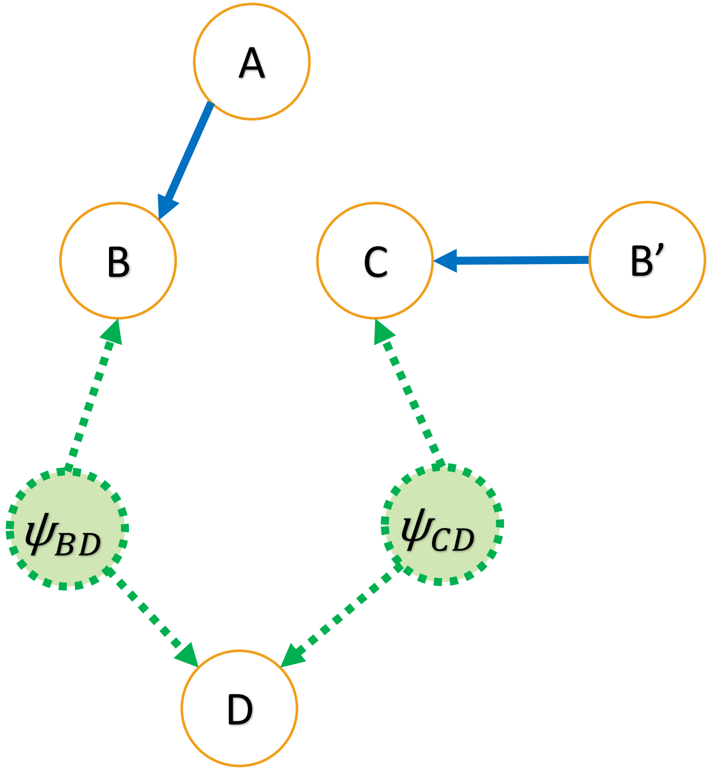
1. $P_{Orig}(A{=}a, B{=}b, C{=}c|X{=}x) = \; P_{Interruption}\,(A{=}a, B{=}b, C{=}c|X{=}x, A'{=}a, B'{=}b')$
2. $P_{Orig}\big(B|do(A{=}a)\big) = \; P_{Interruption}\,(B|A'{=}a)$
3. $P_{Orig}\big(C|do(B{=}b)\big) = \; P_{Interruption}\,(C|B'{=}b)$

Bilocality for constraining do-conditionals #1

Original Scenario

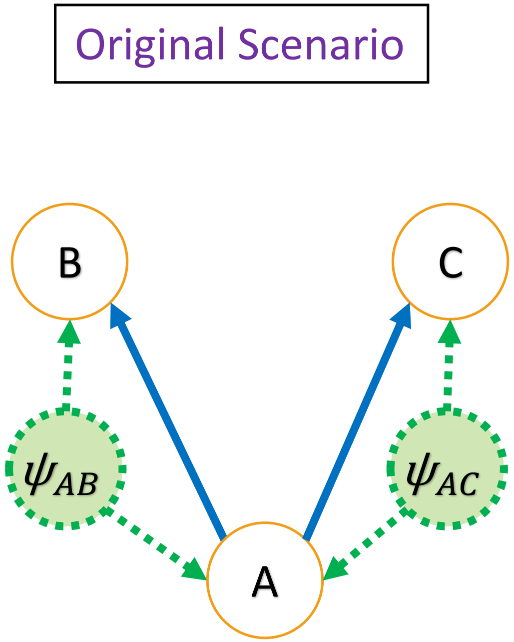


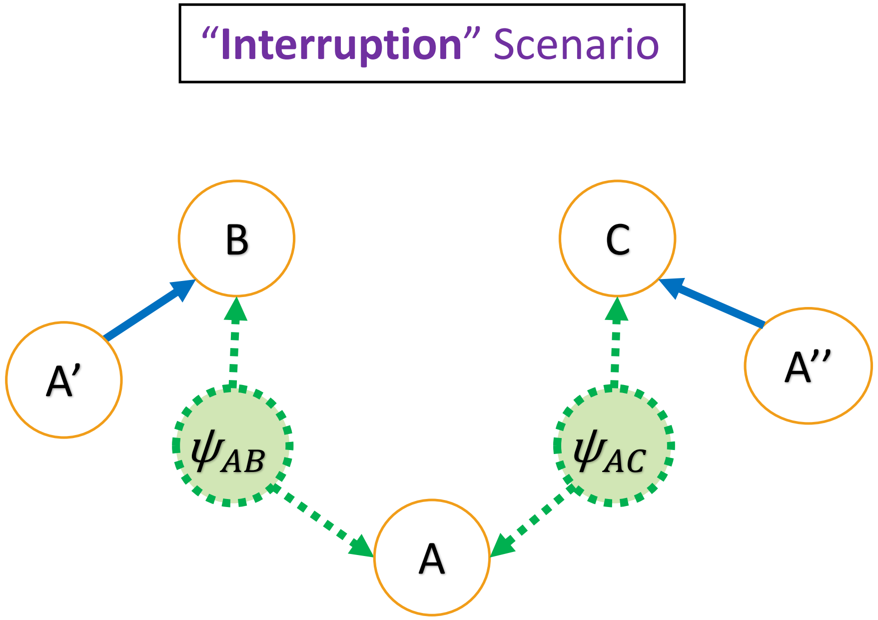
"**Interruption**" Scenario



$$D \perp\!\!\!\perp_{do(B)} A \mid C$$

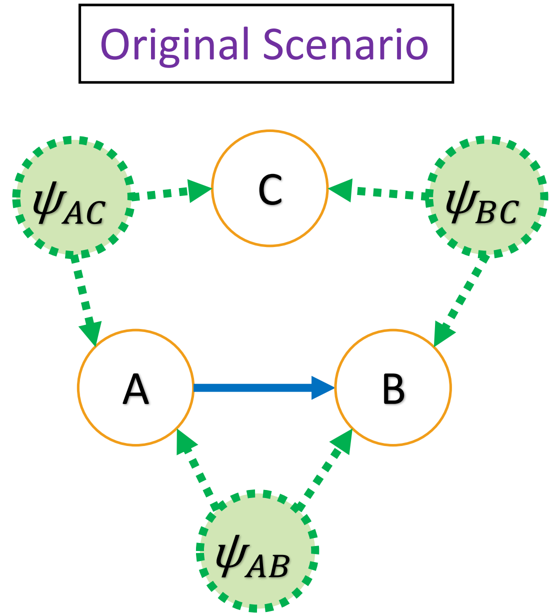
Bilocality for constraining do-conditionals #2

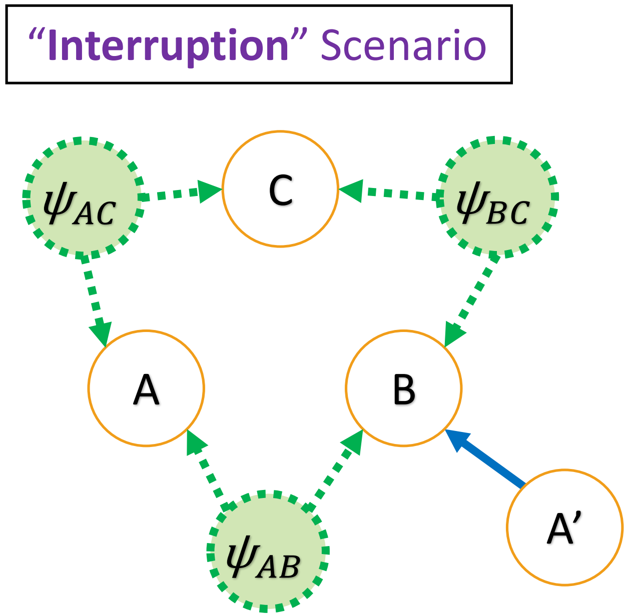




$$B \perp\!\!\!\perp_{do(A)} C$$

Effect Estimation (no constraint implied)





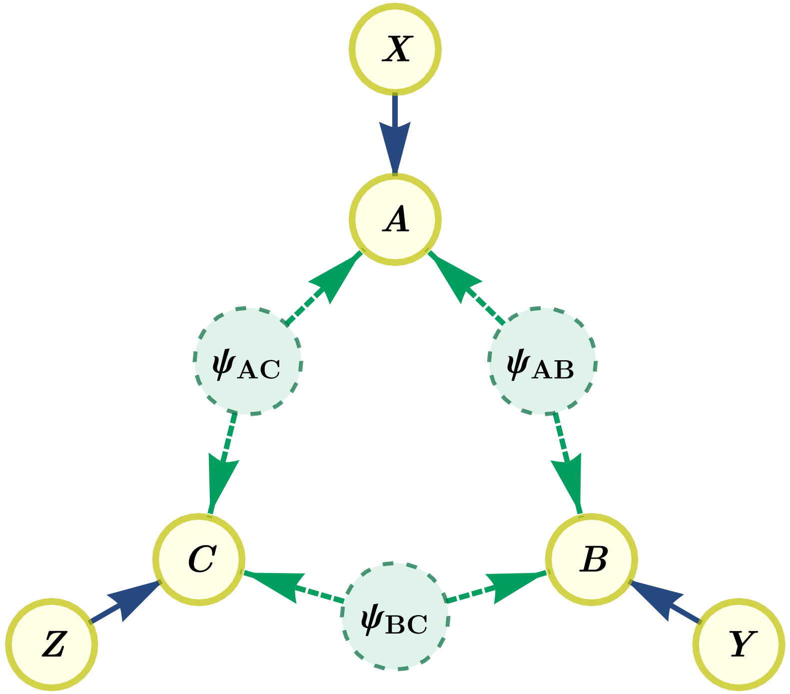
This graph is **saturated** (no inequality constraints). Still, if we observe $P_{Orig}(ABC) = \frac{[000]+[111]}{2}$

then we can conclude that $P_{Orig}(B{=}0|do(A{=}0)) \gg P_{Orig}(B{=}0)$ in **any physical theory**.

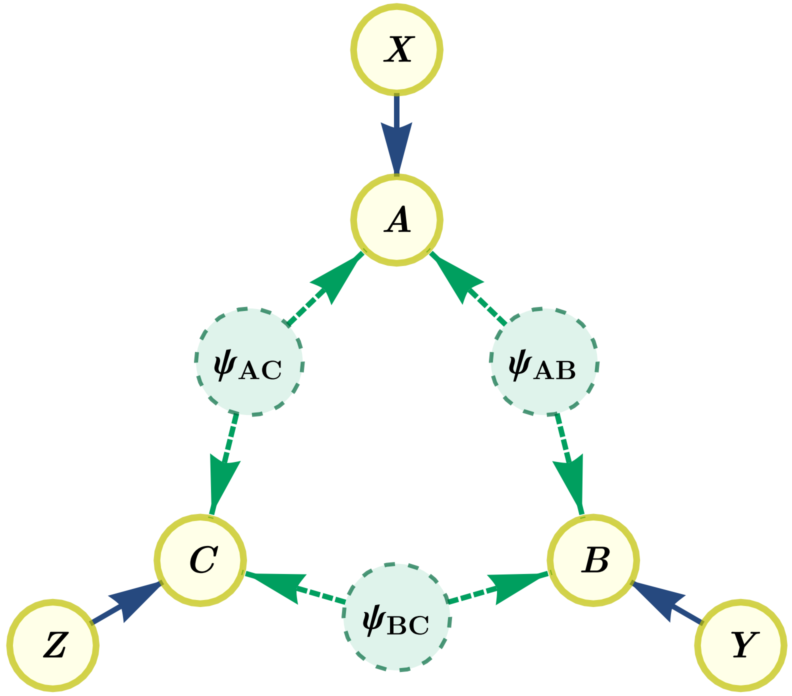
Part Two

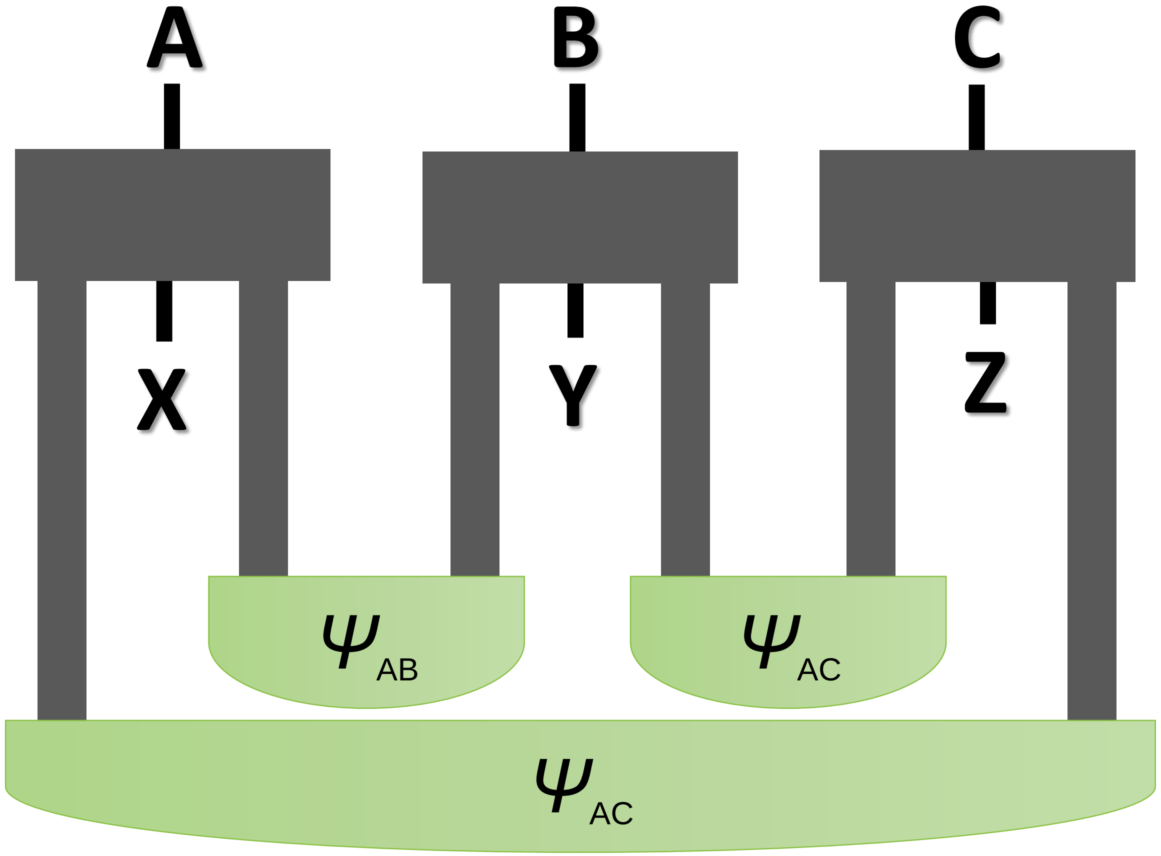
Motivating Quantum Inflation

Quantum Triangle Scenario with Settings

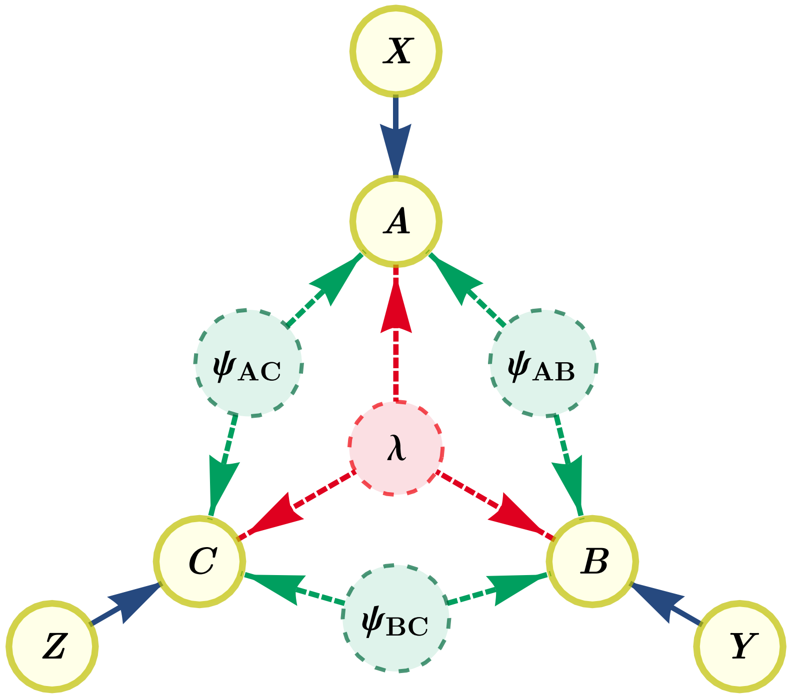


Quantum Channels Picture

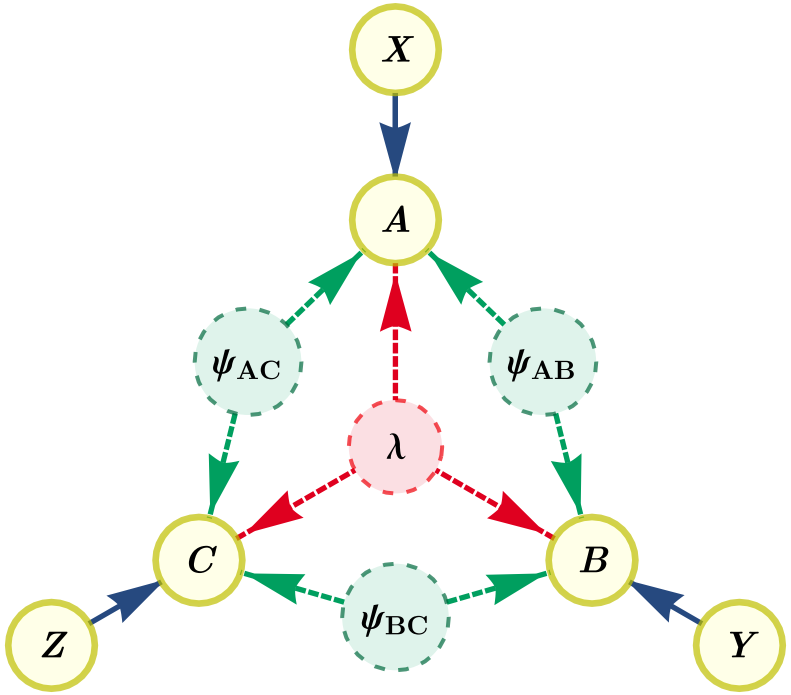


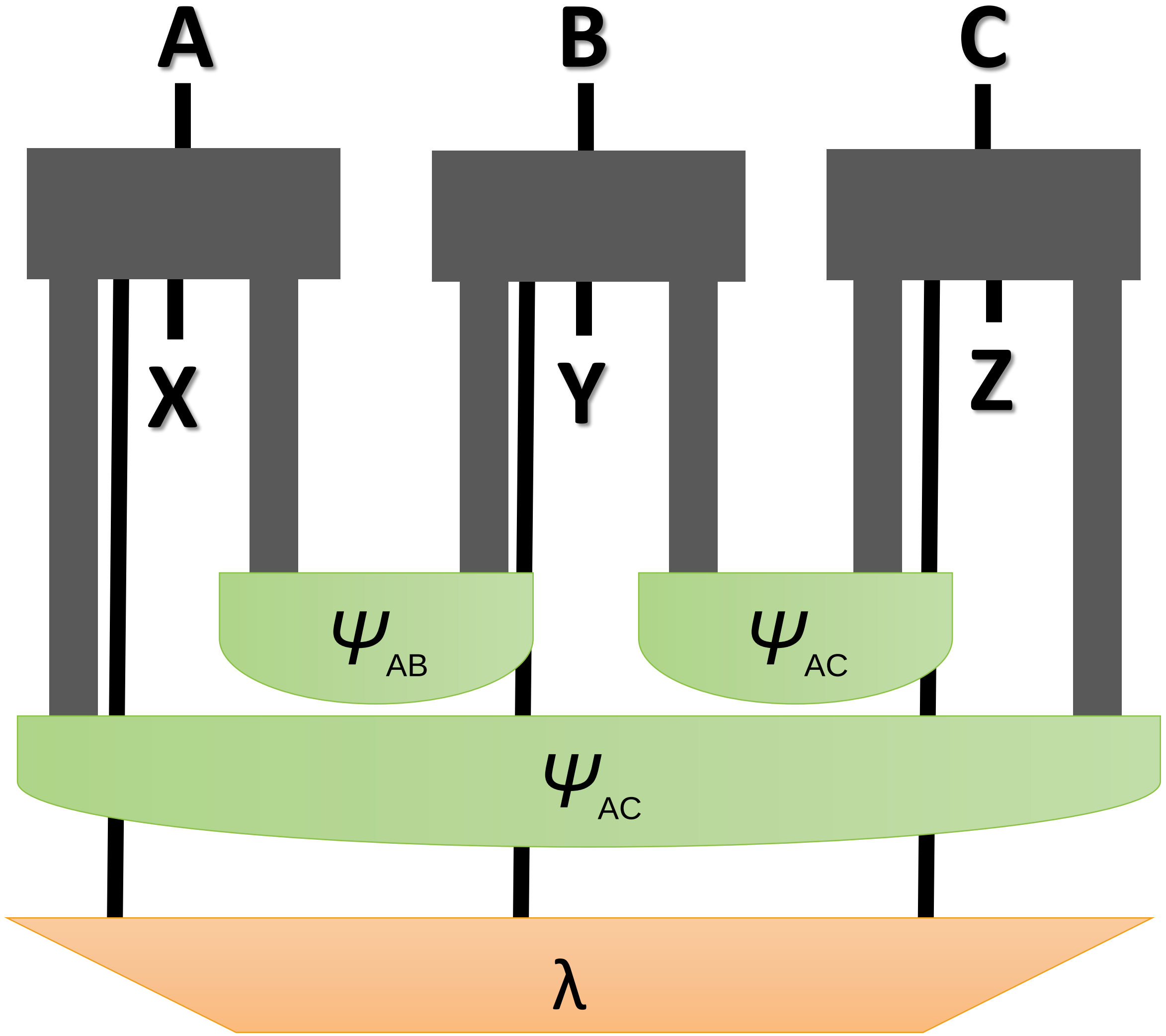


Hybrid Network: (global shared randomness)

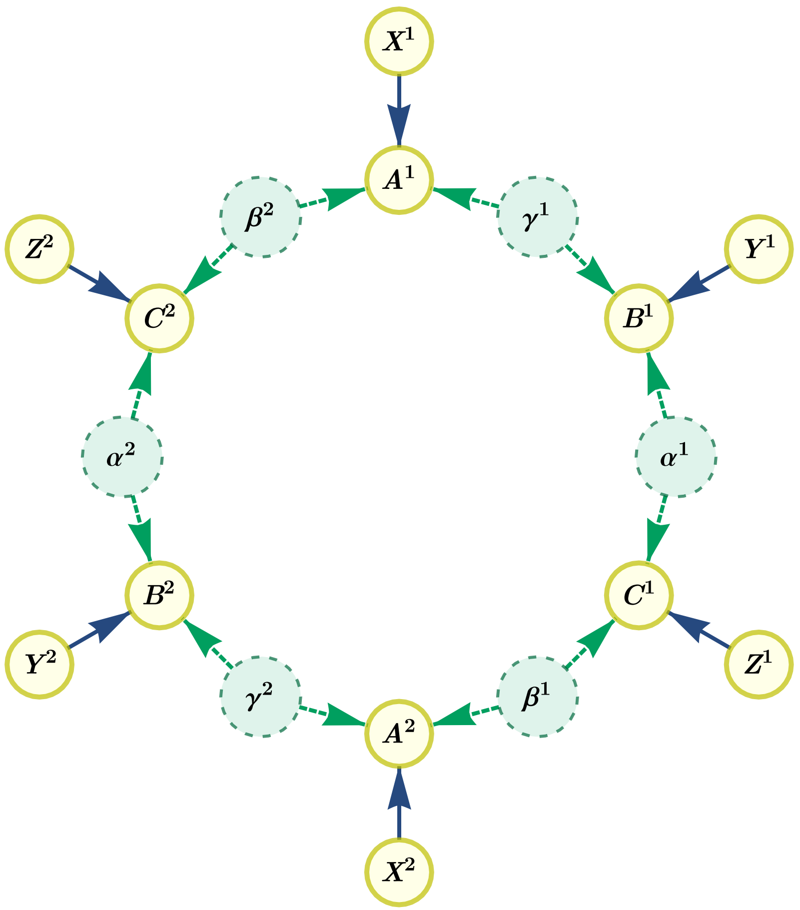


Quantum Channels Picture

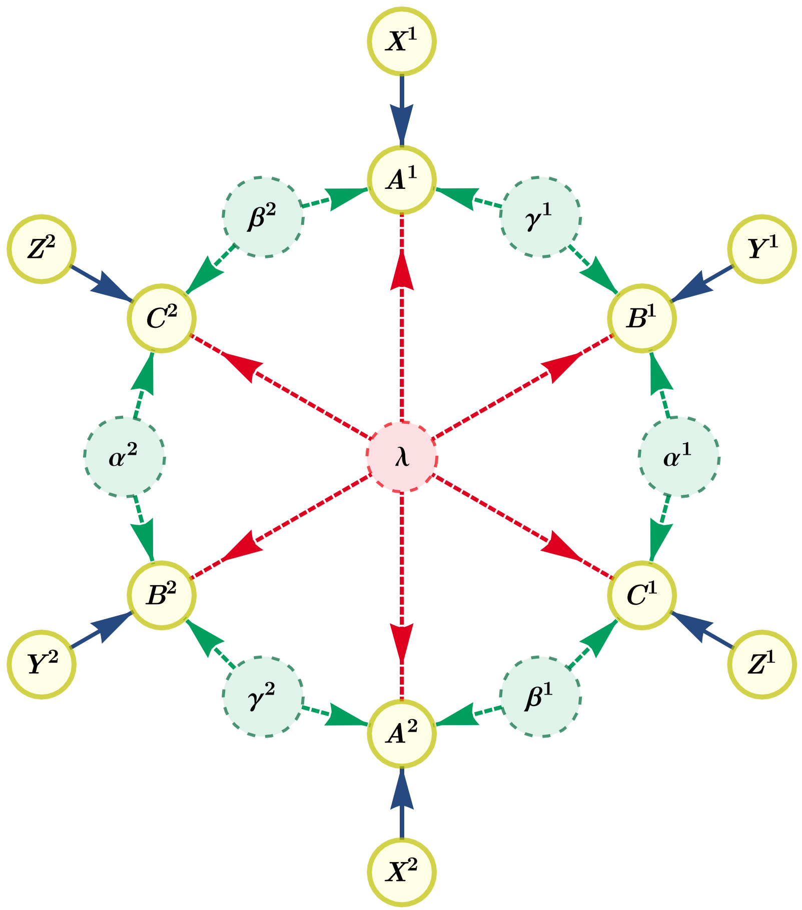




Why Shared Randomness is **HARDER**



$$P(A^1B^1A^2B^2|X^1Y^1X^2Y^2) = P(A^1B^1|X^1Y^1)P(A^2B^2|X^2Y^2)$$



$$P(A^1B^1A^2B^2|X^1Y^1X^2Y^2) \neq P(A^1B^1|X^1Y^1)P(A^2B^2|X^2Y^2)$$

Non-Fanout Inflation References:

- "Theory-independent limits on correlations from generalised Bayesian networks" (Henson, Lal, & Pusey, 2014)
  See Section 4: *"Beyond conditional independence: quantitative bounds on correlations"*
- "The Inflation Technique for Causal Inference with Latent Variables" (**EW**, Spekkens, & Fritz, 2016)
  See Section V-D: *"Implications of the Inflation Technique for Quantum Physics and Generalized Probabilistic Theories"*
- "Constraints on nonlocality in networks from no-signaling and independence" (Gisin et. al., 2019)

A Tale of 3 Boxes and 2 Physical Theories

- Box #8: Not possible in GPT triangle scenario.
- Box #4: Apparently possible in the GPT triangle scenario, but **obviously** not quantum.
- Mermin-GHZ Pseudotelepathy game: Apparently possible in the GPT triangle scenario, but (not obviously!!) not quantum.

"Extremal correlations of the tripartite no-signaling polytope"
(Pironio, Bancal, & Scarani, 2011)

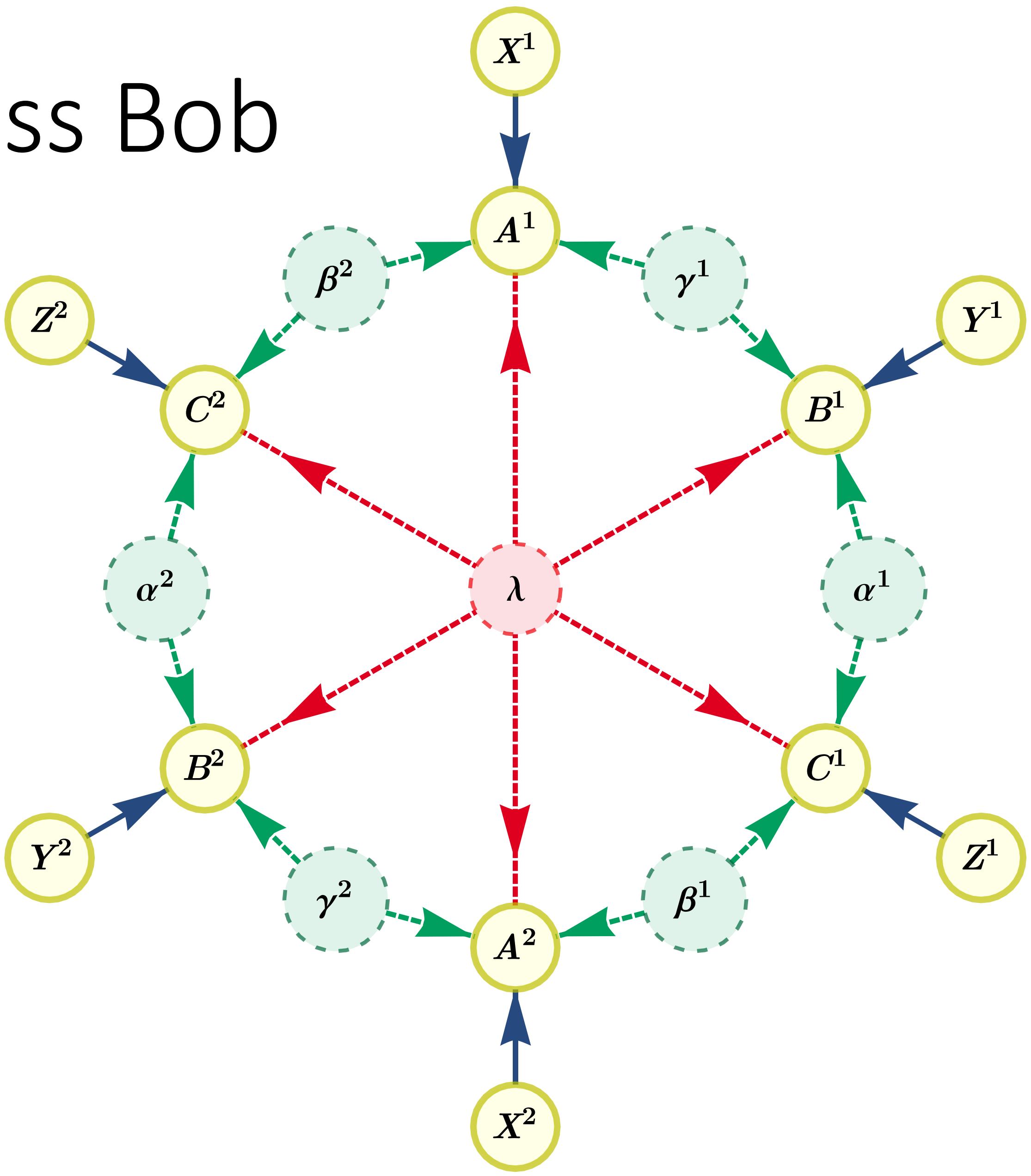
Box #8

$$\langle A_0 B_0 \rangle = +1$$
$$\langle A_0 C_0 \rangle = +1$$
$$\langle A_0 C_1 \rangle = +1$$
$$\langle B_0 C_0 \rangle = +1$$
$$\langle B_0 C_1 \rangle = +1$$
$$\langle A_1 B_1 C_0 \rangle = +1$$
$$\langle A_1 B_1 C_1 \rangle = -1$$

Postquantum, no-signalling tripartite box.

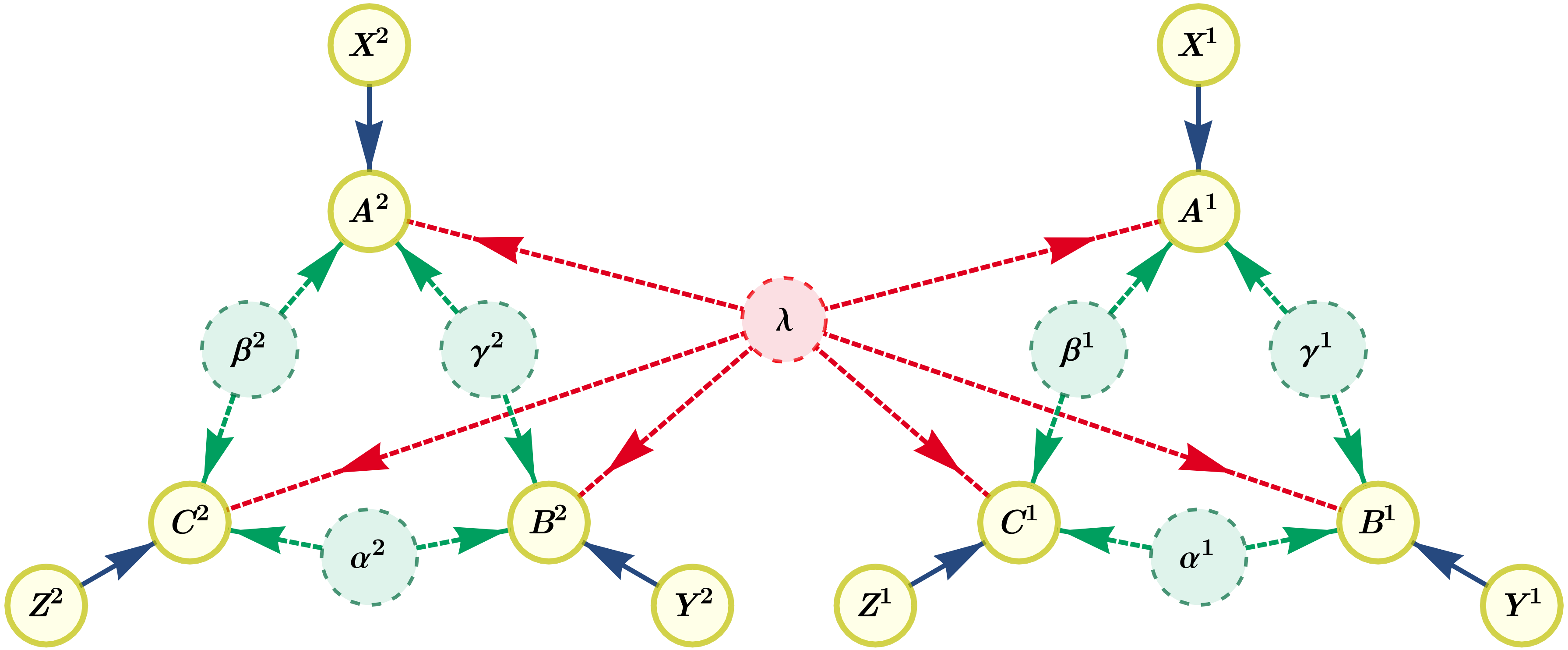
Box #8: Charlie can guess Bob

$$\langle A_{x=0} B_{y=0} \rangle = +1$$
$$\langle A_{x=0} C_{z=0} \rangle = +1$$
...



$C^2_{z=0}$ can **correctly guess** $B^1_{y=0}$ via $B^1_{y=0}$ being correlated with $A^1_{x=0}$ and $A^1_{x=0}$ being correlated with $C^2_{z=0}$

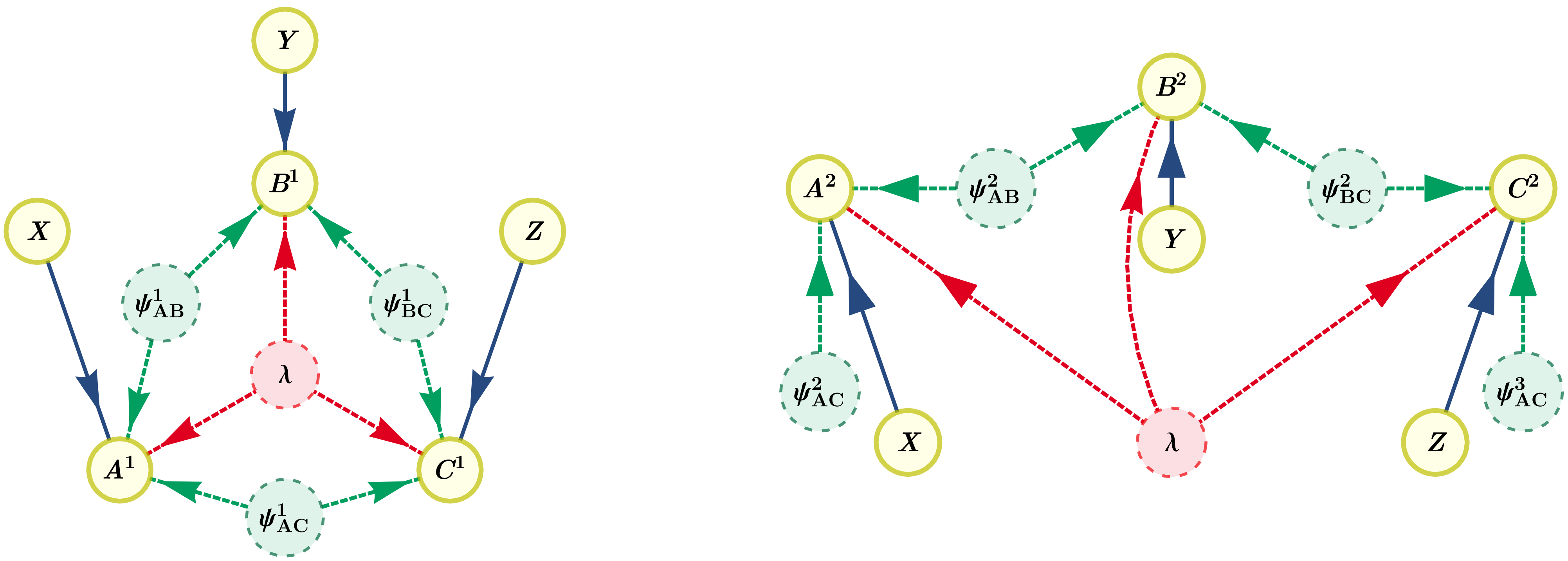
Extremality prohibits 4th party guessing…



$B_{1|y=0}\ C_{2|z=0}$ must be a **product distribution**
if $A_{1|x=0}\ B_{1|y=0}\ C_{2|z=0}$ is an extremal tripartite NS box.

CONTRADICTION.

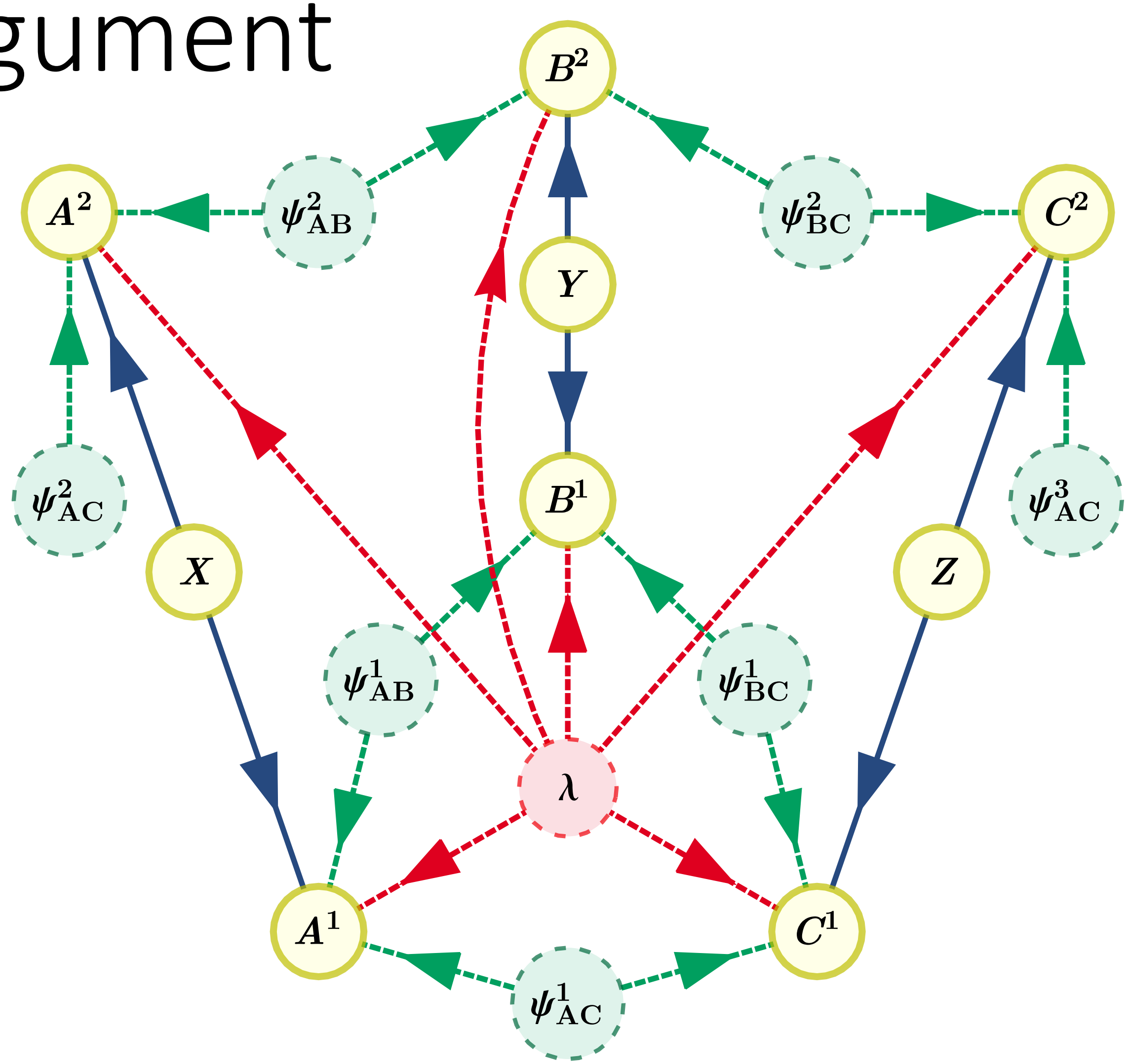
Box #8: Alternative Argument



$$P(A^1B^1|XY) = P(A^2B^2|XY)$$
$$P(B^1C^1|YZ) = P(B^2C^2|YZ)$$

Box #8: Alternative Argument

$\langle A_{x=0} B_{y=0} \rangle = +1$
$\langle B_{y=0} C_{z=0} \rangle = +1$
...



$$P(A^1 B^1 | XY) = P(A^2 B^2 | XY)$$
$$P(B^1 C^1 | YZ) = P(B^2 C^2 | YZ)$$
$$P(A^2 C^2 | XZ) = P(A^1 C^2 | XZ)$$

n-way extremality vs. *(n-1)*-way correlation

- Many tripartite extremal NS boxes are evidently incompatible with the (GPT) triangle scenario.

- Quantum version: Extremality of a box in the 3-way **quantum** correlations set **conflicts** with two-pairs of bipartite correlation

- FYI: There exists a **noisy variant** of Box #8 which is **GPT-triangle incompatible** but admits quantum realization using **3-way entanglement**.

Non PR-ness Proofs:

- "Popescu-Rohrlich Correlations as a Unit of Nonlocality"
  (Barrett & Pironio, 2005)
  See Theorem 2: *(5-cycle graph-state correlations cannot be simulated via PR boxes)*

- "Test to separate quantum theory from non-signaling theories"
  (Chao & Reichardt, 2017)

- "Separating pseudo-telepathy games and two-local theories"
  (Mathieu & Mhalla, 2018)

Box #4

$$\langle A_0 B_1 \rangle = +1$$
$$\langle B_0 C_1 \rangle = +1$$
$$\langle C_0 A_1 \rangle = +1$$
$$\langle A_0 B_0 C_0 \rangle = +1$$
$$\langle A_1 B_1 C_1 \rangle = -1$$

Postquantum, no-signalling tripartite box.

No **chain** of bipartite correlation terms!

Box #4

$$\langle A_0 B_1 \rangle = +1$$
$$\langle B_0 C_1 \rangle = +1$$
$$\langle C_0 A_1 \rangle = +1$$
$$\langle A_0 B_0 C_0 \rangle = +1$$
$$\langle A_1 B_1 C_1 \rangle = -1$$

Box #4 appears to be **GPT-realizable** in the triangle scenario!

This, despite the fact that it **cannot be realized via any wiring of PR boxes***.

(Obviously quantum incompatible.)

*Per "Feats, Features and Failures of the PR-box" (Scarani, 2005)

Does Box #4 admit a Triangle GPT realization?

- Stefano thinks so, but still an open question!

- Wirings are weaker that GPT entangled measurements. See: "Couplers for non-locality swapping" (Linden & Brunner, 2009) and "Generalizations of Boxworld" (Janotta, 2012)

- See also: "Information-Causality and Extremal Tripartite Correlations" (Yang et. al. 2012)
  See Section IV: *"Class #4: Extremal No-Signalling Correlations Satisfying and Bipartite Criterion"*

Example: Mermin-GHZ Pseudotelepathy

$$\langle A_0 B_0 C_1 \rangle = +1$$
$$\langle A_0 B_1 C_0 \rangle = +1$$
$$\langle A_1 B_0 C_0 \rangle = +1$$
$$\langle A_1 B_1 C_1 \rangle = -1$$

$$|\psi_{ABC}\rangle = \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

Setting "0" $= \sigma_Y$

Setting "1" $= \sigma_X$

Mermin-GHZ success w/ 3-way entanglement

Mermin-GHZ can be simulated with a PR box

$$\langle A_0 B_0 C_1 \rangle = +1$$
$$\langle A_0 B_1 C_0 \rangle = +1$$
$$\langle A_1 B_0 C_0 \rangle = +1$$
$$\langle A_1 B_1 C_1 \rangle = -1$$

(Just have Charlie output +1 deterministically for both settings.)

Mermin-GHZ failure w/ 2-way entanglement

$$P_{Mermin,\boldsymbol{v}}(abc|xyz) = \begin{cases} \frac{1}{8} & x+y+z=0 \mod 2 \\ \left(1+\boldsymbol{v}(-1)^{a+b+c}\right)/8 & x+y+z=1 \\ \left(1-\boldsymbol{v}(-1)^{a+b+c}\right)/8 & x+y+z=3 \end{cases}, \quad \boldsymbol{v} \leq \sqrt{5/8}$$

$$\langle A_0B_0C_1\rangle + \langle A_0B_1C_0\rangle + \langle A_1B_0C_0\rangle - \langle A_1B_1C_1\rangle \leq \sqrt{10}$$

See **Quantum Inflation** technique – talk by Toni Acin, poster of Alex Pozas-Kerstjens

Mermin-GHZ References:

- “Quantum mysteries revisited” (Mermin, 1990)
- “Recasting Mermin's multi-player game into the framework of pseudo-telepathy” (Brassard et. al., 2005)
- “On the power of non-local boxes” (Broadbent & Méthot, 2006)

Review of Motivating Questions

QUESTION: What is a tripartite **quantum correlation** which **cannot** be realized if the parties share **2-way GPT resources** (and 3-way classical shared randomness?

TECHNIQUE: No-signalling inequalities (from hexagon ring inflation of the triangle).

SOLUTION: Noisy version of tripartite extremal NS Box #8.

QUESTION: What is a tripartite quantum correlations which **could** be realized if the parties share **2-way GPT resources** but **not** if they only share **2-way quantum resources**?

TECHNIQUE: **Quantum Inflation**.

SOLUTION: The Mermin-GHZ nonlocal box (psuedotelepathy, GHZ-state self-test.)

Previous relevant (but incomplete) ideas

- *"The Inflation Technique for Causal Inference with Latent Variables"*
  EW, Robert W. Spekkens, Tobias Fritz arXiv:1609.00672
  Deficiency: Cannot distinguish quantum from GPT.

- *"Information-theoretic implications of quantum causal structures"*
  Rafael Chaves, Christian Majenz, David Gross arXiv:1407.3800
  See also *"Analysing causal structures in generalised probabilistic theories"*
  Mirjam Weilenmann, Roger Colbeck arXiv:1812.04327
  Deficiency: Insensitive. Does not rule out W in quantum triangle.

- *"Bounding the sets of classical and quantum correlations in networks"*
  Alejandro Pozas-Kerstjens, Rafael Rabelo, Łukasz Rudnicki, Rafael Chaves, Daniel Cavalcanti, Miguel Navascues, Antonio Acín arXiv:1904.08943
  Deficiency: Leverages independence, so not applicable to quantum triangle.

Summary

- Edges originating from non-root observed variables:
  INTERRUPTION
  (followed by traditional quantum constraining)

- Multiple root quantum nodes:
  NON-FANOUT INFLATION
  (holds for any physical theory)
  followed by
  QUANTUM INFLATION
  (if need be)

Thank You