

Quantifying Bell: The Resource Theory of Nonclassicality of Common-cause Boxes

Robert Spekkens
Perimeter Institute

joint work with:

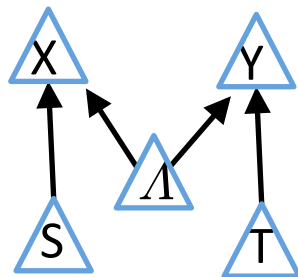
Elie Wolfe, David Schmid, Ana Belén Sainz, and Ravi Kunjwal



Causality in a quantum world
Capri, Sept 12, 2019

Causal Model

Causal
Structure



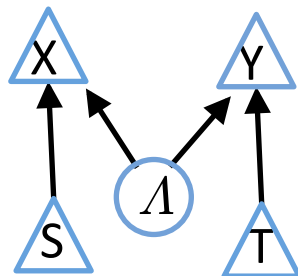
Causal-Statistical
Parameters

 P_S P_T P_Λ $P_{X|\Lambda S}$ $P_{Y|\Lambda T}$

$$P_{XYST\Lambda} = P_{X|S\Lambda} P_{Y|T\Lambda} P_\Lambda P_S P_T$$

Causal Model

Causal
Structure



Causal-Statistical
Parameters

P_S

P_T

P_Λ

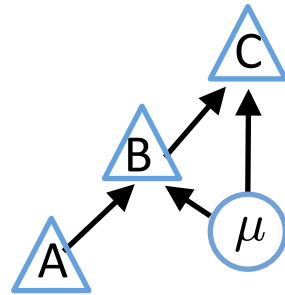
$P_{X|\Lambda S}$

$P_{Y|\Lambda T}$

$$P_{XYST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda} P_S P_T$$

Causal Model

Causal
Structure



Causal-Statistical
Parameters

$$P_{C|B\mu}$$

$$P_{B|A\mu}$$

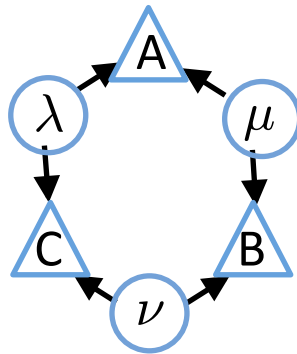
$$P_{\mu}$$

$$P_A$$

$$P_{ABC} = \sum_{\mu} P_{C|\mu B} P_{B|A\mu} P_A P_{\mu}$$

Causal Model

Causal
Structure



Causal-Statistical
Parameters

$$P_{A|\mu\lambda}$$

$$P_{B|\mu\nu}$$

$$P_{C|\nu\lambda}$$

$$P_{\mu}$$

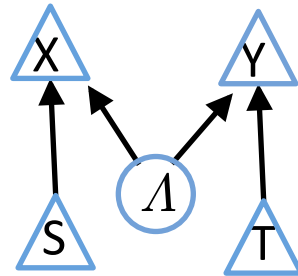
$$P_{\nu}$$

$$P_{\lambda}$$

$$P_{ABC} = \sum_{\mu\nu\lambda} P_{A|\mu\lambda} P_{B|\mu\nu} P_{C|\nu\lambda} P_{\mu} P_{\nu} P_{\lambda}$$

Causal Model

Causal
Structure



Causal-Statistical
Parameters

$$P_S$$

$$P_T$$

$$P_\Lambda$$

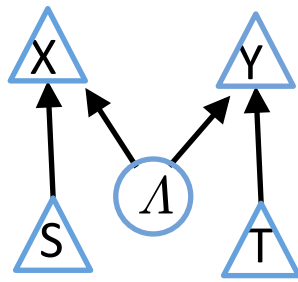
$$P_{X|\Lambda S}$$

$$P_{Y|\Lambda T}$$

$$P_{XYST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda} P_S P_T$$

A distribution P on observed variables is **compatible** with a DAG if there exists parameters on that DAG that generate P

What are the conditions for compatibility?



$$P_{\Lambda}$$

$$P_{X|\Lambda S}$$

$$P_{Y|\Lambda T}$$

Conditional independence
relations (equality constraints)

$$X \perp T | S$$

$$Y \perp S | T$$

$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

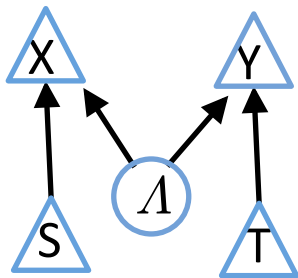
Inequality constraints

$$\frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|01)$$

$$\frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4} \sum_{x \neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

J.S. Bell, Physics 1, 195 (1964)

Clauser, Horne, Shimony and Holte, Phys. Rev. Lett. 23, 880 (1967)



$$P_{\Lambda}$$

$$P_{X|\Lambda S}$$

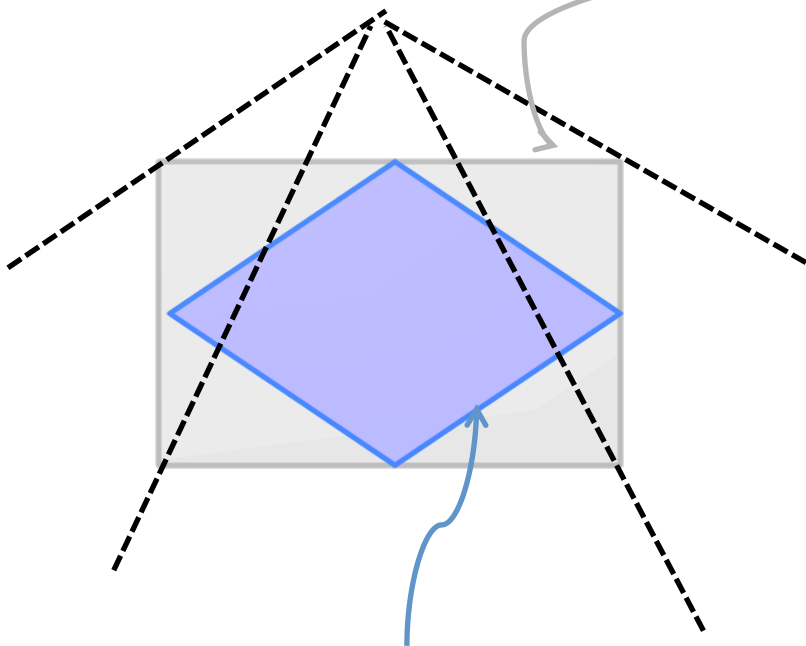
$$P_{Y|\Lambda T}$$

$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

Equality
constraints

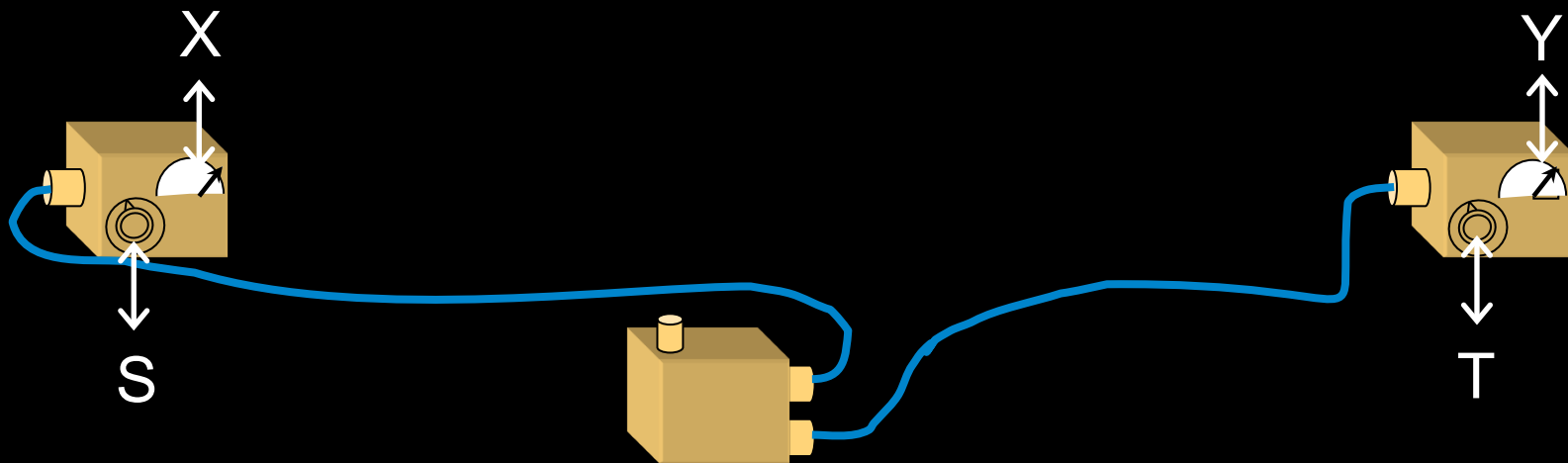
$$X \perp T|S$$

$$Y \perp S|T$$



Inequality
constraint

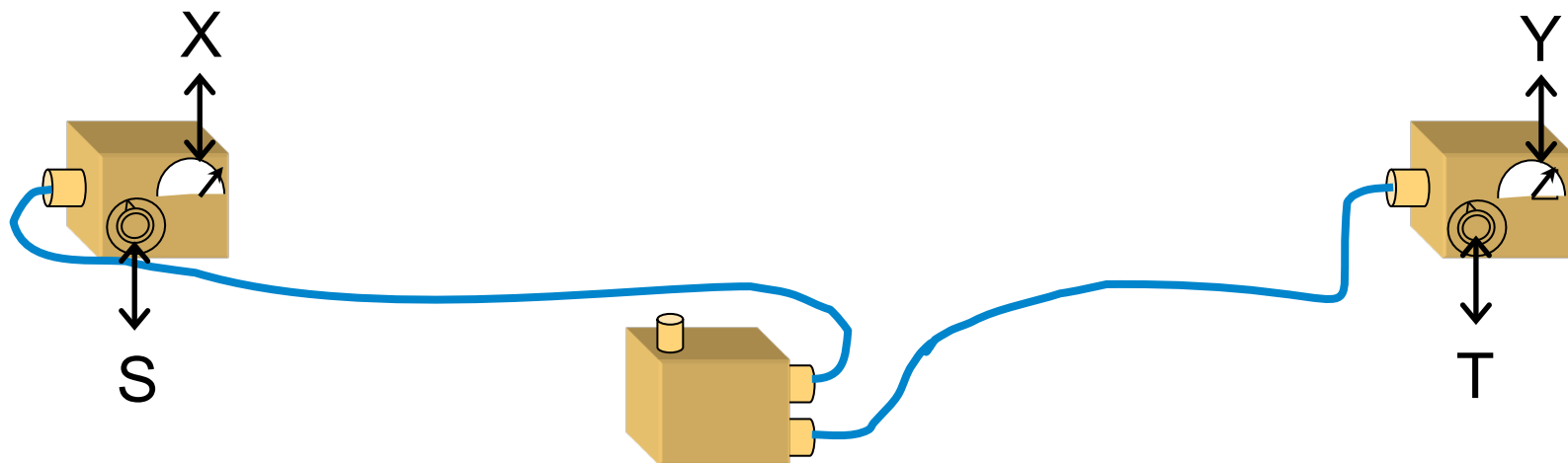
$$\frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|01) + \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4} \sum_{x \neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$



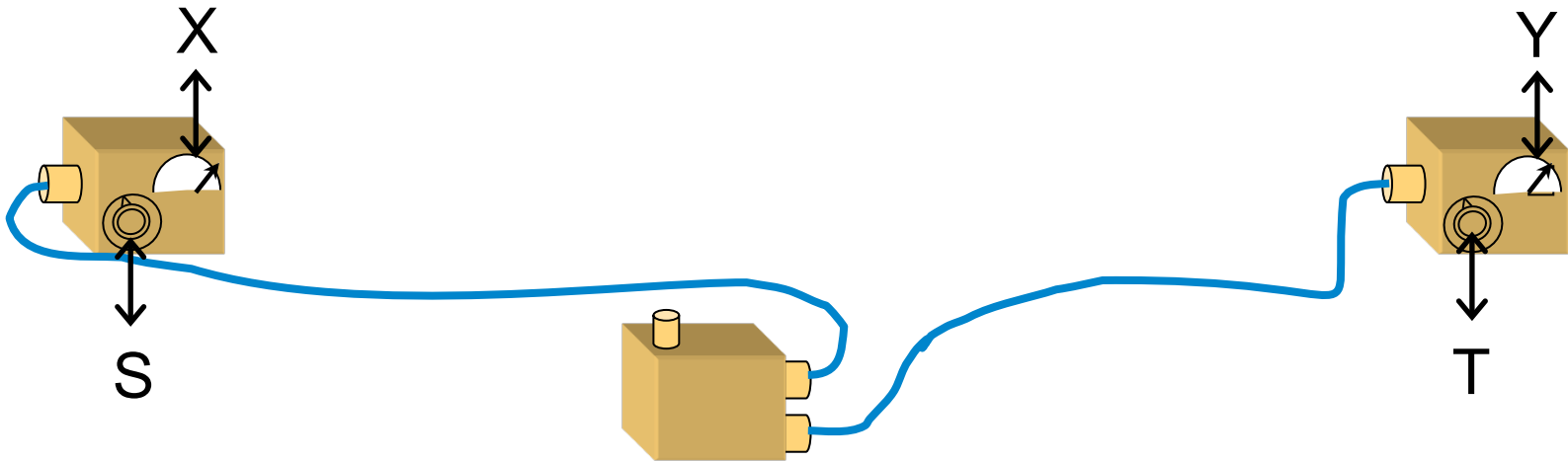
$$P(X,Y|S,T)$$

| | X=0, Y=0 | X=0, Y=1 | X=1, Y=0 | X=1, Y=1 |
|-------------|-------------|-------------|-------------|-------------|
| S=0, T=0 | 0.427 | 0.073 | 0.073 | 0.427 |
| S=0, T=1 | 0.427 | 0.073 | 0.073 | 0.427 |
| S=1, T=0 | 0.427 | 0.073 | 0.073 | 0.427 |
| S=1, T=1 | 0.073 | 0.427 | 0.427 | 0.073 |

Quantum correlations can *violate* the Bell inequalities



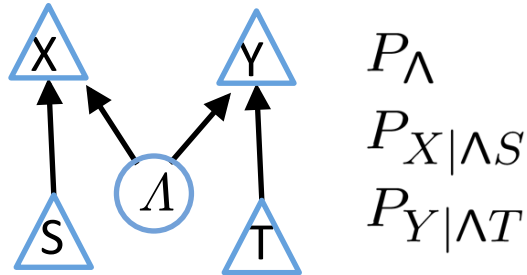
But we still need to provide a causal explanation of the experimental statistics



Relativity theory \rightarrow causal influences, not just signals, are subluminal

Also, causal models that invoke superluminal causation but are no-signalling must have parameters that are fine-tuned
(Wood and RWS, New J. Phys. 17, 033002 (2015))

Classical Causal Models



$$P_{\Lambda}$$

$$P_{X|\Lambda S}$$

$$P_{Y|\Lambda T}$$

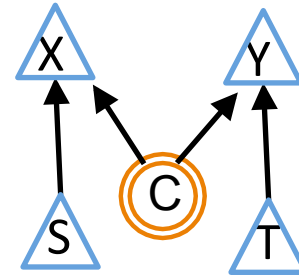
$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

$$P_{XY|ST}(xy|st)$$

$$= \sum_{\lambda} P_{X|S\Lambda}(x|s\lambda) P_{Y|T\Lambda}(y|t\lambda) P_{\Lambda}(\lambda)$$

Satisfies the Bell inequalities

Quantum Causal Models



$$\rho_C$$

$$\rho_{X|SC}$$

$$\rho_{Y|TC}$$

where

$$[\rho_{X|SC}, \rho_{Y|TC}] = 0$$

$$\rho_{XY|ST} = \text{Tr}_C(\rho_{X|SC} \rho_{Y|TC} \rho_C)$$

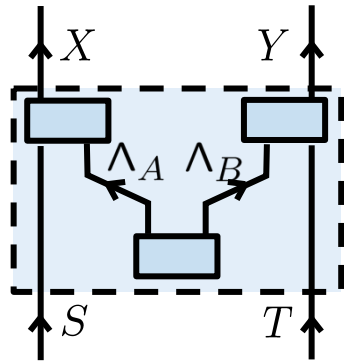
$$P_{XY|ST}(xy|st)$$

$$= \langle x | \langle y | \langle s | \langle t | \rho_{XY|ST} | x \rangle | y \rangle | s \rangle | t \rangle$$

Violates the Bell inequalities

Leifer, RWS, Phys. Rev. A 88, 052130 (2013)
 Allen, Barrett, Horsman, Lee, RWS, PRX 7,
 031021 (2017)

Classical Causal Models



$$P_{\Lambda_A \Lambda_B}$$

$$P_{X|\Lambda_A S}$$

$$P_{Y|\Lambda_B T}$$

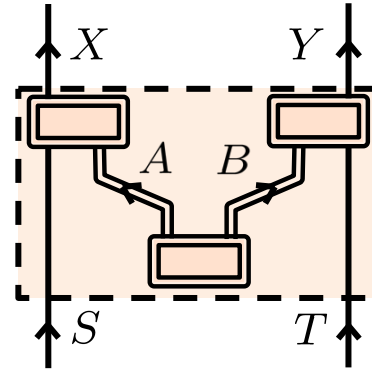
$$P_{XY|ST} = \sum_{\Lambda_A \Lambda_B} P_{X|S \Lambda_A} P_{Y|T \Lambda_B} P_{\Lambda_A \Lambda_B}$$

$$P_{XY|ST}(xy|st)$$

$$= \sum_{\lambda_A \lambda_B} P_{X|S \Lambda_A}(x|s \lambda_A) P_{Y|T \Lambda_B}(y|t \lambda_B) P_{\Lambda_A \Lambda_B}(\lambda_A \lambda_B)$$

Satisfies the Bell inequalities

Quantum Causal Models



$$\rho_{AB}$$

$$\rho_{X|SA}$$

$$\rho_{Y|TB}$$

$$\rho_{XY|ST} = \text{Tr}_{AB}((\rho_{X|SA} \otimes \rho_{Y|TB}) \rho_{AB})$$

$$P_{XY|ST}(xy|st)$$

$$= \langle x | \langle y | \langle s | \langle t | \rho_{XY|ST} | x \rangle | y \rangle | s \rangle | t \rangle$$

Violates the Bell inequalities

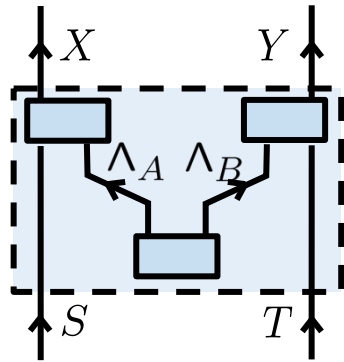
Henson, Lal and Pusey, New J. Phys. 16, 113043 (2014)

Fritz, Comm. Math. Phys. 341, 391 (2016)

Costa and Shrapnel, New J. Phys. 18, 063032 (2016)

Allen, Barrett, Horsman, Lee, RWS, Phys. Rev. X 7, 031021 (2017)

Classical Causal Models



$$P_{\Lambda_A \Lambda_B}$$

$$P_{X|S \Lambda_A}$$

$$P_{Y|T \Lambda_B}$$

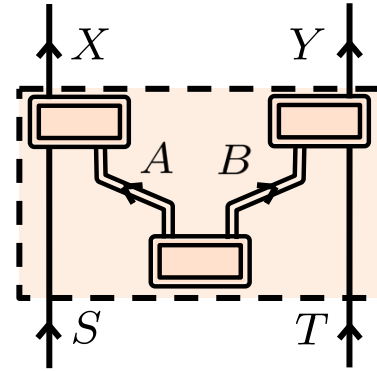
$$P_{XY|ST} = \sum_{\Lambda_A \Lambda_B} P_{X|S \Lambda_A} P_{Y|T \Lambda_B} P_{\Lambda_A \Lambda_B}$$

$$P_{XY|ST}(xy|st)$$

$$= \sum_{\lambda_A \lambda_B} P_{X|S \Lambda_A}(x|s \lambda_A) P_{Y|T \Lambda_B}(y|t \lambda_B) P_{\Lambda_A \Lambda_B}(\lambda_A \lambda_B)$$

Satisfies the Bell inequalities

Quantum Causal Models



$$\rho_{AB}$$

$$E_{x|s}^A$$

$$E_{y|t}^B$$

$$P_{XY|ST}(xy|st) = \text{Tr}_{AB}((E_{x|s}^A \otimes E_{y|t}^B) \rho_{AB})$$

Violates the Bell inequalities

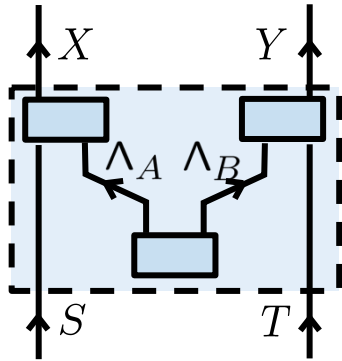
Henson, Lal and Pusey, New J. Phys. 16, 113043 (2014)

Fritz, Comm. Math. Phys. 341, 391 (2016)

Costa and Shrapnel, New J. Phys. 18, 063032 (2016)

Allen, Barrett, Horsman, Lee, RWS, Phys. Rev. X 7, 031021 (2017)

Classical Causal Models



$$P_{\Lambda_A \Lambda_B}$$

$$P_{X|S \Lambda_A}$$

$$P_{Y|T \Lambda_B}$$

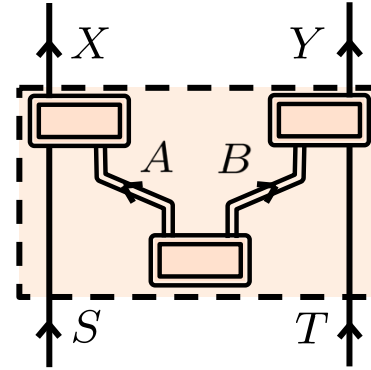
$$P_{XY|ST} = \sum_{\Lambda_A \Lambda_B} P_{X|S \Lambda_A} P_{Y|T \Lambda_B} P_{\Lambda_A \Lambda_B}$$

$$P_{XY|ST}(xy|st)$$

$$= \sum_{\lambda_A \lambda_B} P_{X|S \Lambda_A}(x|s \lambda_A) P_{Y|T \Lambda_B}(y|t \lambda_B) P_{\Lambda_A \Lambda_B}(\lambda_A \lambda_B)$$

Satisfies the Bell inequalities

GPT Causal Models



$$\mathbf{s}^{AB} \in \mathbb{R}^{d_A} \otimes \mathbb{R}^{d_B}$$

$$\mathbf{r}_{x|s}^A \in \mathbb{R}^{d_A}$$

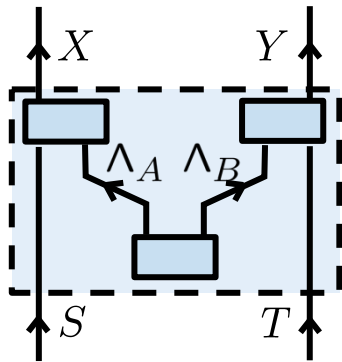
$$\mathbf{r}_{y|t}^B \in \mathbb{R}^{d_B}$$

$$P_{XY|ST}(xy|st) = (\mathbf{r}_{x|s}^A \otimes \mathbf{r}_{y|t}^B) \cdot \mathbf{s}^{AB}$$

Violates the Bell inequalities
maximally

Henson, Lal and Pusey, New J. Phys. 16, 113043 (2014)
Fritz, Comm. Math. Phys. 341, 391 (2016)

Classical Causal Models



$$P_{\Lambda_A \Lambda_B}$$

$$P_{X|\Lambda_A S}$$

$$P_{Y|\Lambda_B T}$$

Conditional independence
relations

$$X \perp T | S$$

$$Y \perp S | T$$

$$P_{XY|ST} = \sum_{\Lambda_A \Lambda_B} P_{X|S \Lambda_A} P_{Y|T \Lambda_B} P_{\Lambda_A \Lambda_B}$$

A Bell inequality

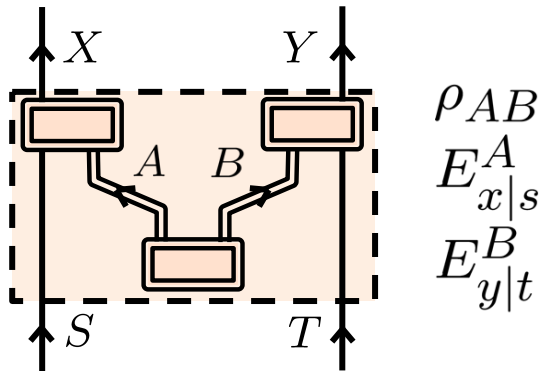
$$\frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|01)$$

$$\frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4} \sum_{x \neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

J.S. Bell, Physics 1, 195 (1964)

Clauser, Horne, Shimony and Holte, Phys. Rev. Lett. 23, 880 (1967)

Quantum Causal Models



Conditional independence
relations

$$X \perp T | S$$

$$Y \perp S | T$$

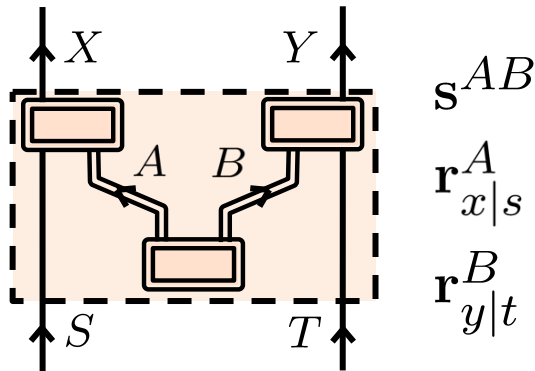
$$P_{XY|ST}(xy|st) = \text{Tr}_{AB}((E_{x|s}^A \otimes E_{y|t}^B) \rho_{AB})$$

A Tsirelson inequality

$$\begin{aligned} & \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|01) \\ & \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4} \sum_{x \neq y} P_{XY|ST}(xy|11) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}} \end{aligned}$$

Tsirelson, Lett. Math. Phys. 4, 93 (1980),
Popescu and Rohrlich, Found. Phys. 24, 379 (1994).

GPT Causal Models



Conditional independence
relations

$$X \perp T | S$$

$$Y \perp S | T$$

$$P_{XY|ST}(xy|st) = (r_{x|s}^A \otimes r_{y|t}^B) \cdot s^{AB}$$

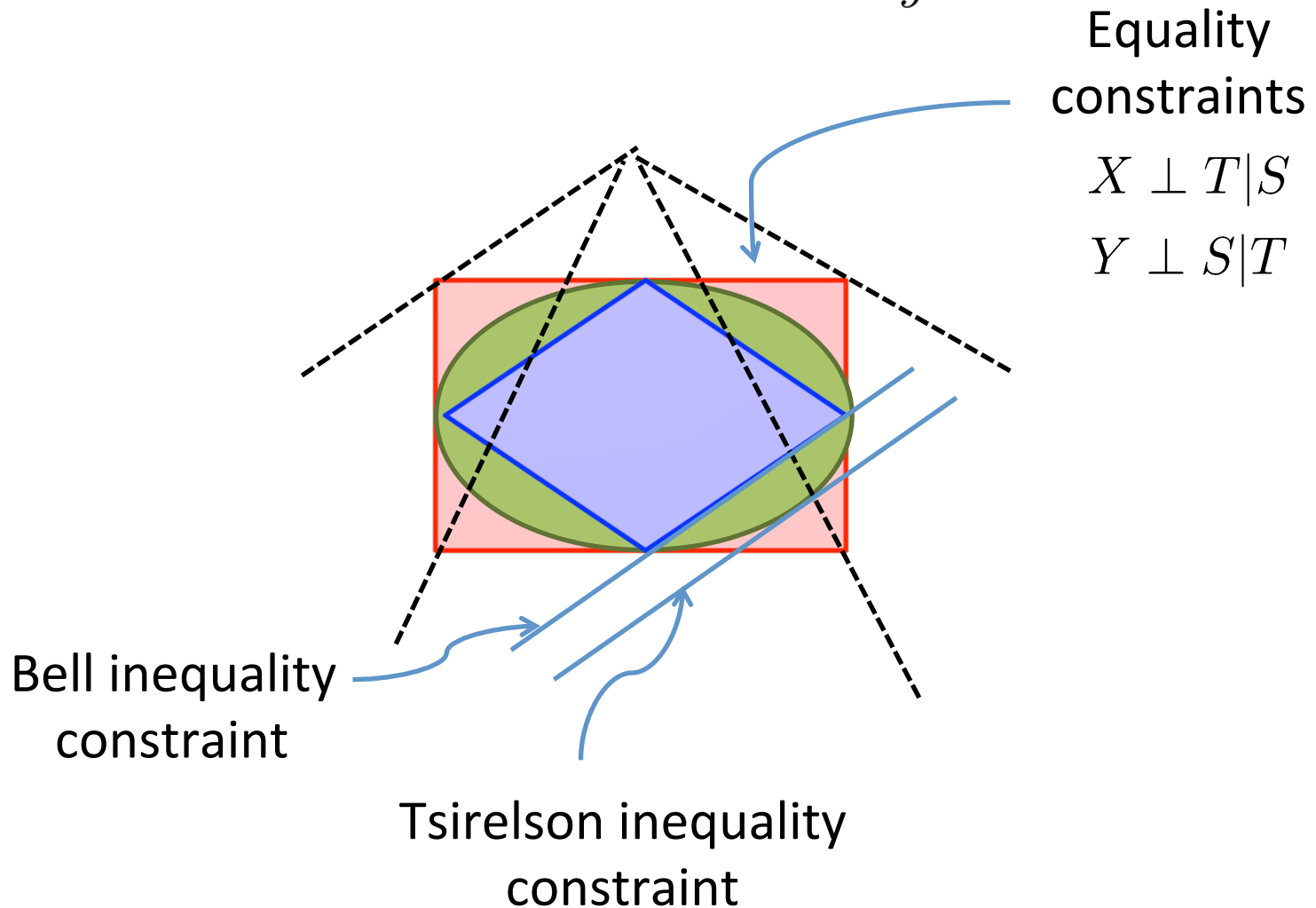
No nontrivial inequalities

$$\begin{aligned} & \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|01) \\ & \frac{1}{4} \sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4} \sum_{x \neq y} P_{XY|ST}(xy|11) \leq 1 \end{aligned}$$

Tsirelson, Lett. Math. Phys. 4, 93 (1980),
Popescu and Rohrlich, Found. Phys. 24, 379 (1994).

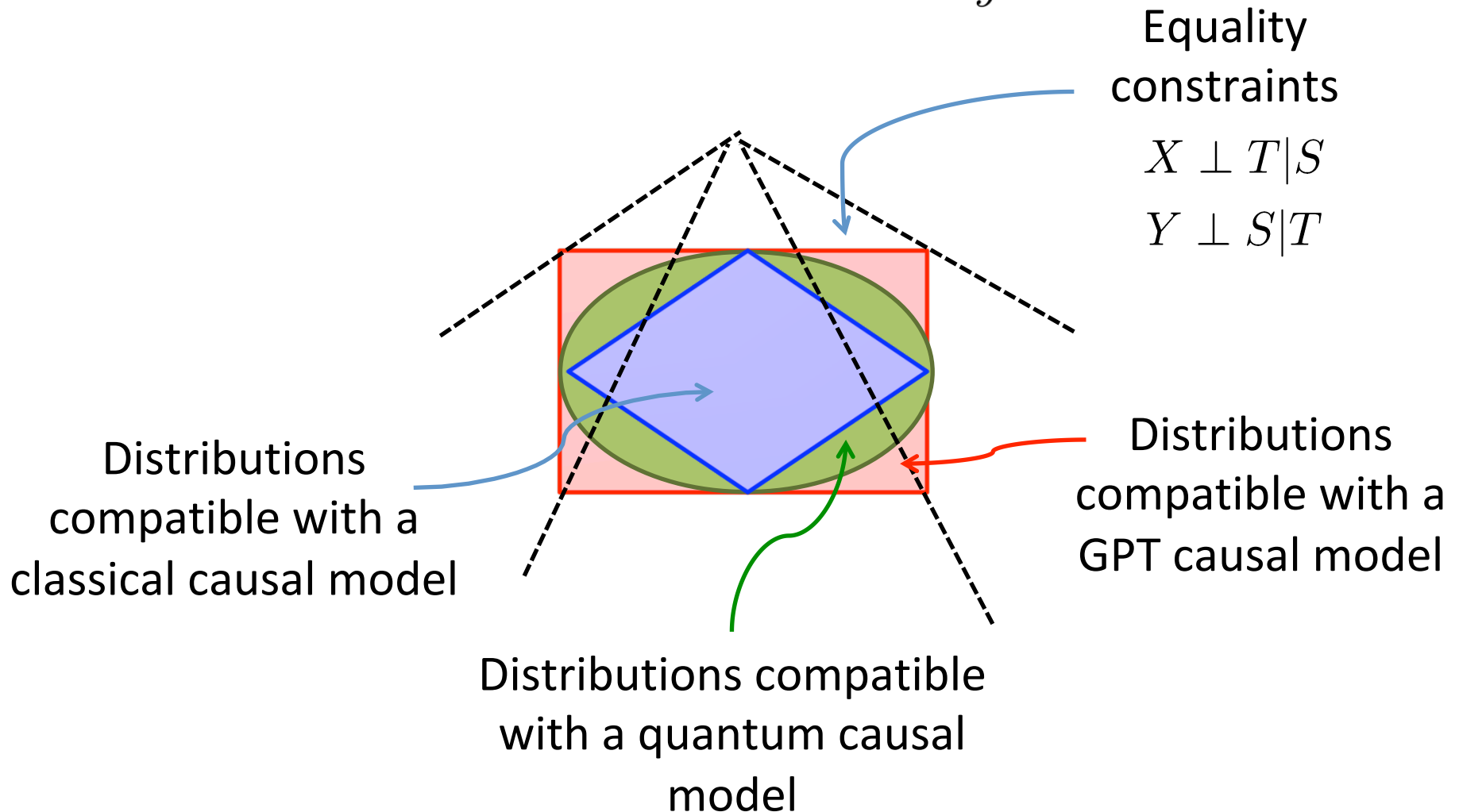
Space of compatible probability distributions

$$\vec{R} = \left(P_{XY|ST}(xy|st) \right)_{xyst}$$



Space of compatible probability distributions

$$\vec{R} = \left(P_{XY|ST}(xy|st) \right)_{xyst}$$



Inequalities describing compatibility with a classical causal model for causal structures different from that of the Bell scenario

Pearl, UAI proceedings (1995)

Janzing and Beth, IJQI 4, 347 (2006)

Steudel and Ay, arXiv:1010.5720

Fritz, New J. Phys. 14, 103001 (2012)

Chaves and Fritz, PRA 85 (2012)

Branciard, Rosset, Gisin, Pironio, PRA 85, 3 (2012)

Henson, Lal, Pusey, New J. Phys. 16, 113043 (2014)

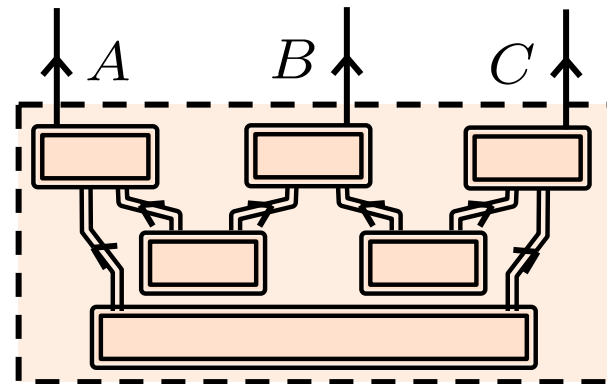
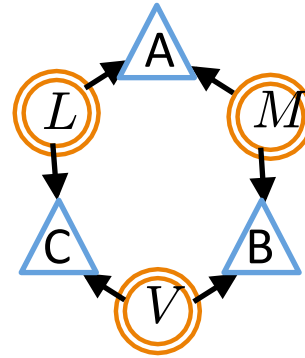
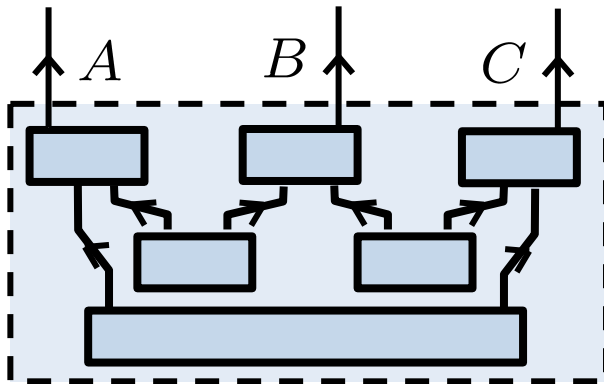
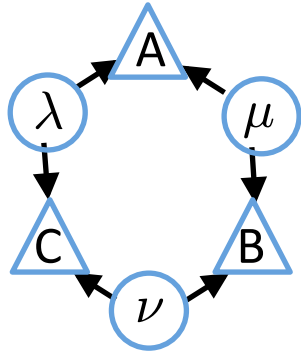
Chaves, Luft, Gross, New J. Phys. 16, 043001 (2014)

Wolfe, Fritz, RWS, J. Causal Inference 7 (2019) [arXiv:1609.00672]

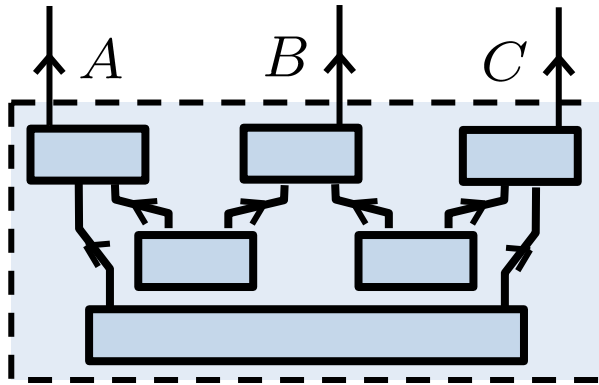
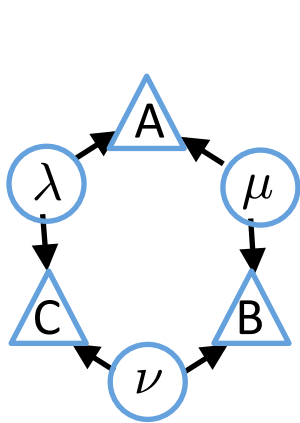
Fraser and Wolfe, Phys. Rev A 98, 022113 (2018)

Chaves, Carvacho, Agresti, Di Giulio, Aolita, Giacomini, and Sciarrino, Nat. Phys. 14, 291 (2018)

The triangle scenario w/ 4-valued outcomes



Classical Causal Models



No Conditional
independence relations
among observed variables

Inequality constraints:

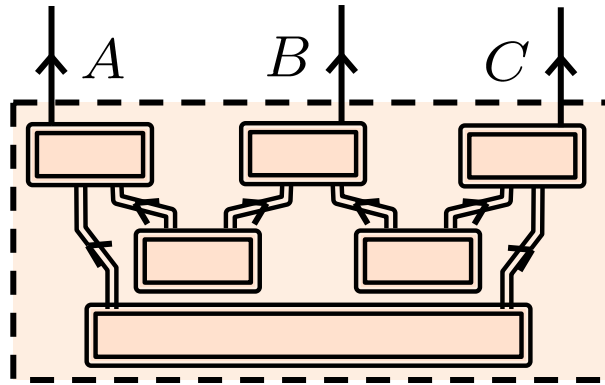
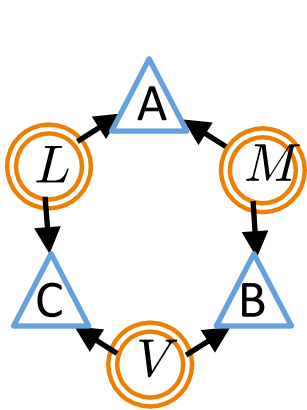
$$I(A : B) + I(A : C) \leq H(A)$$

$$\begin{aligned} &+P_{A_l B_l}(11) - P_{A_l B_l C_l C_r}(1111) + P_{A_l B_l}(00)P_{C_l C_r}(11) + P_{C_l C_r}(01)P_{C_l C_r}(10) \\ &- P_{C_l C_r}(11)P_{A_l A_r B_l B_r C_l C_r}(000000) - P_{C_l C_r}(11)P_{A_l A_r B_l B_r C_l C_r}(010100) \\ &- P_{C_l C_r}(10)P_{A_l A_r B_l B_r C_l C_r}(001001) - P_{C_l C_r}(10)P_{A_l A_r B_l B_r C_l C_r}(011101) \leq 0 \\ &- P_{C_l C_r}(01)P_{A_l A_r B_l B_r C_l C_r}(100110) - P_{C_l C_r}(01)P_{A_l A_r B_l B_r C_l C_r}(110010) \\ &+ P_{C_l C_r}(00)P_{A_l A_r B_l B_r C_l C_r}(101111) + P_{C_l C_r}(00)P_{A_l A_r B_l B_r C_l C_r}(111011) \end{aligned}$$

Fritz, New J. Phys. 14, 103001 (2012)

Fraser and Wolfe, Phys. Rev A 98, 022113 (2018)

Quantum Causal Models



No Conditional
independence relations
among observed variables

Inequality constraints:

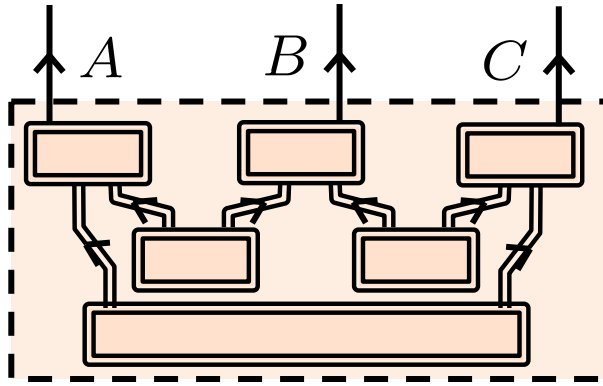
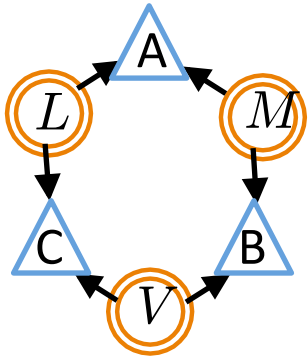
$$I(A : B) + I(A : C) \leq H(A)$$

Inequalities that can in principle be
derived from quantum inflation

Henson, Lal, Pusey, New J. Phys. 16, 113043 (2014)

Wolfe, Pozas-Kerstjens, Grinberg, Rosset, Acin, Navascues, in preparation

GPT Causal Models



No Conditional
independence relations
among observed variables

Inequality constraints:

$$I(A : B) + I(A : C) \leq H(A)$$

Henson, Lal, Pusey, New J. Phys. 16, 113043 (2014)

$$\vec{R} = \left(P_{XY|ST}(xy|st) \right)_{xyst}$$


Nonclassicality provides advantages for
information-processing tasks

This is a motivation for **characterizing** the boundary between
classical and nonclassical

But such a characterization is not enough:
We also need to **quantify** the nonclassicality

Resource Theory

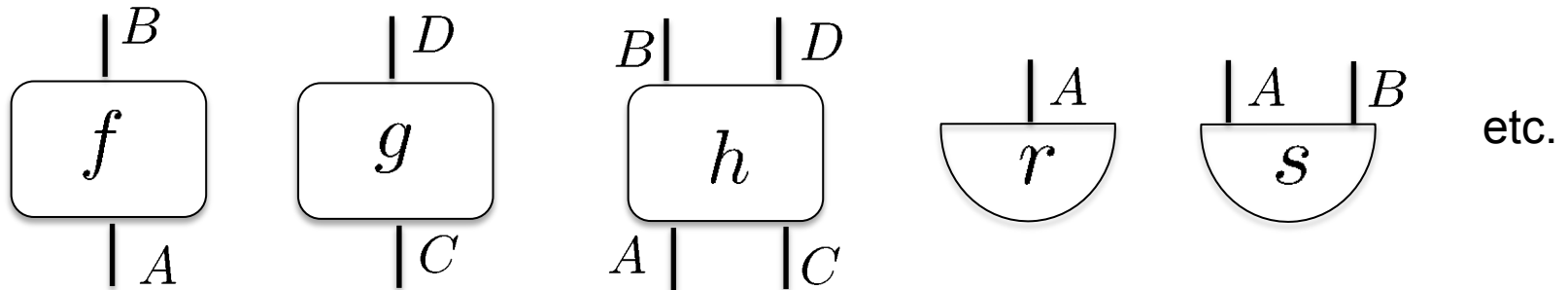
Preliminaries

System types A, B, C, \dots

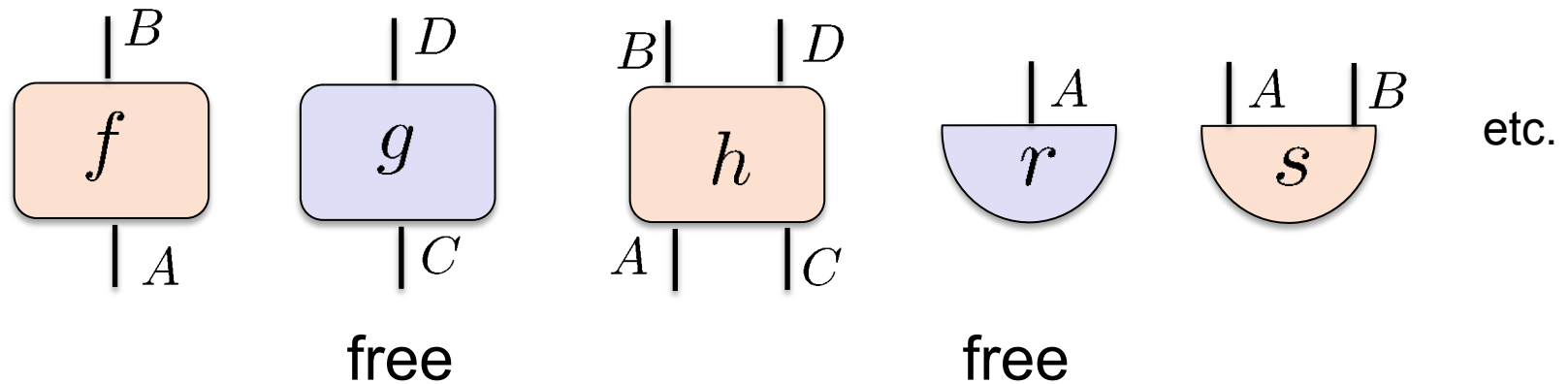
Process theory T : (including the trivial system)

Processes f, g, h, r, s, \dots

Closed under parallel and sequential composition



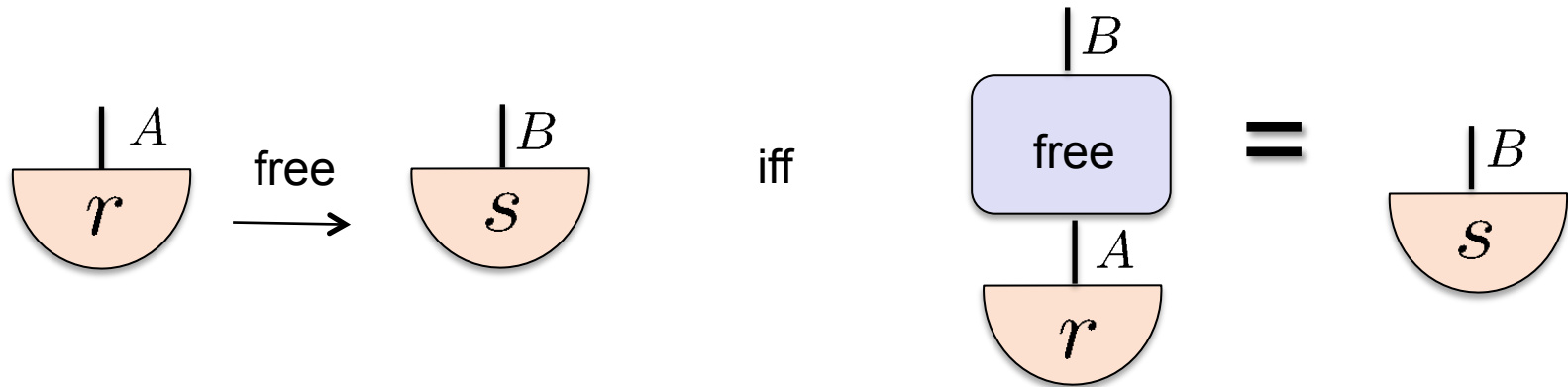
System types A, B, C, \dots
 Process theory T : (including the trivial system)
 Processes f, g, h, r, s, \dots
 Closed under parallel and sequential composition



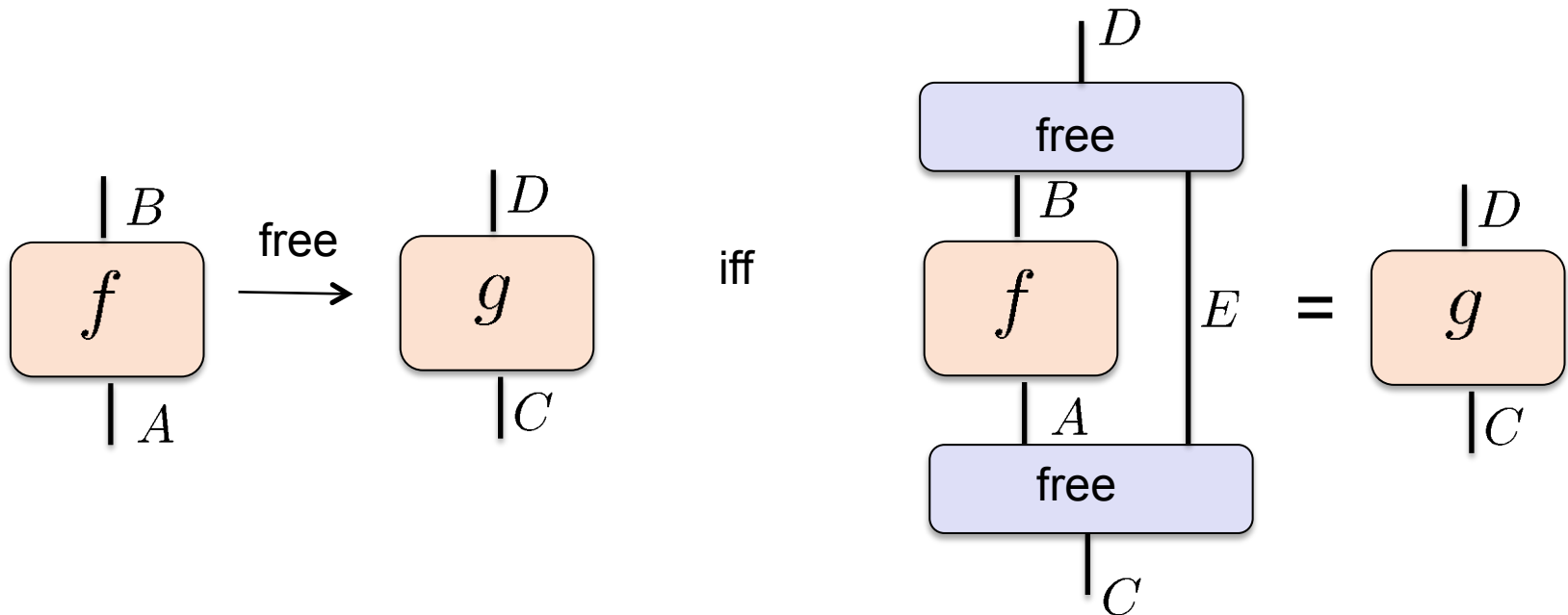
Subtheory of “free” processes T_{free} :

A resource theory is a
 partitioned process theory
 (T, T_{free})

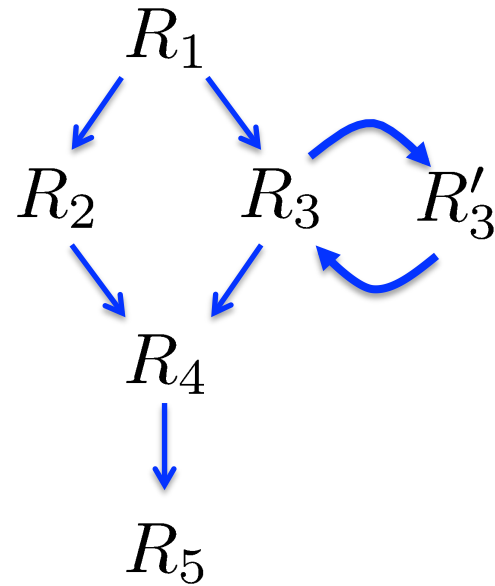
Conversion of state resources:



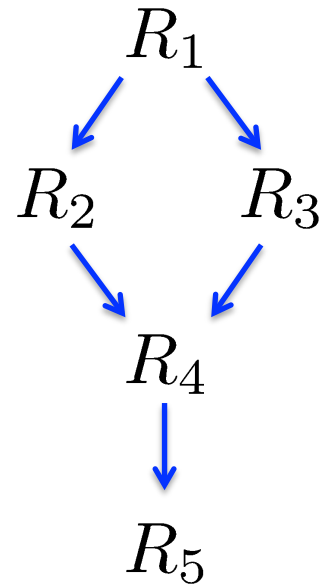
Conversion of channel resources:



Conversion relations induce a preorder of resources



Quotienting equivalences, one gets a partial order of resources



The nature of the pre-order teaches us about the resource

Properties of a pre-order

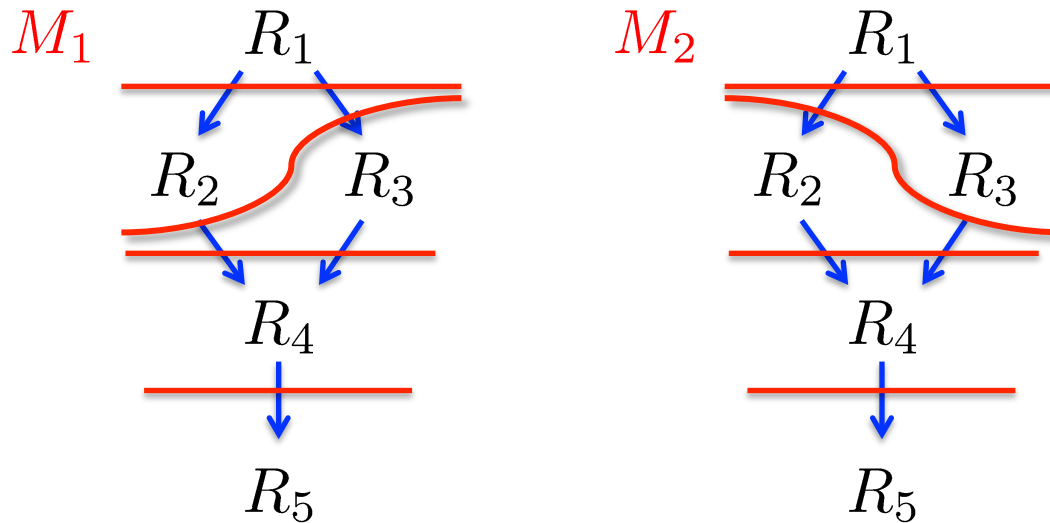
- Totally pre-ordered (no incomparable elements) or not
- weak (incomparability relation is transitive) or not
- Height (cardinality of the largest chain)
- Width (cardinality of the largest antichain)
- Locally finite (finite number of inequivalent elements between any two ordered elements) or not

Measures of a resource

Def'n: A function M from resources to the reals is a **resource monotone** if

$$\forall R_1, R_2 : R_1 \xrightarrow{\text{free}} R_2 \quad \Rightarrow \quad M(R_1) \geq M(R_2)$$

Equivalently, M must respect the partial order



If it is not a total order, there cannot be
“one measure to rule them all”

A family of monotones $\{ M_i \}_i$ is **complete** if it completely characterizes the pre-order,

$$\forall R_1, R_2 : R_1 \xrightarrow{free} R_2 \quad \Leftrightarrow \quad \forall i : M_i(R_1) \geq M_i(R_2)$$

Yield and Cost constructions

Given any function f (monotone or not) together with a set \mathbf{S} of resources

Yield construction $M[f\text{-yield}, \mathbf{S}](R) :=$

$$\max_{R^* \in \mathbf{S}} \{f(R^*) \text{ s.t. } R \longmapsto R^*\}$$

Cost construction $M[f\text{-cost}, \mathbf{S}](R) :=$

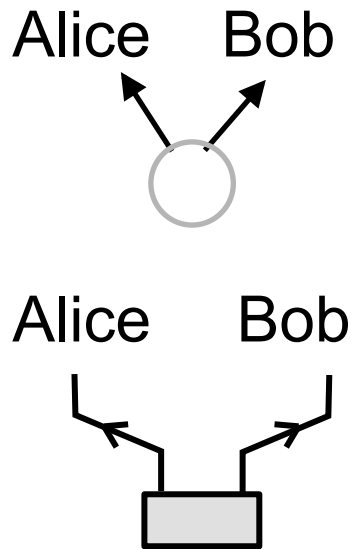
$$\min_{R^* \in \mathbf{S}} \{f(R^*) \text{ s.t. } R^* \longmapsto R\}$$

($+\infty$ if no resource in \mathbf{S} can go to R)

Note: R is of any type

How to define the
resource theory of
nonclassicality of
correlations for
network causal
structures

2-party network with a common cause structure



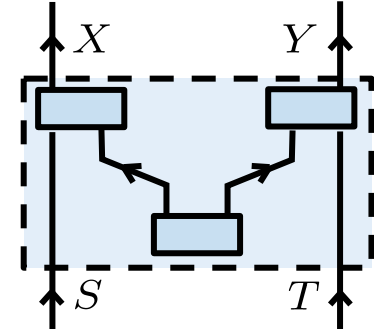
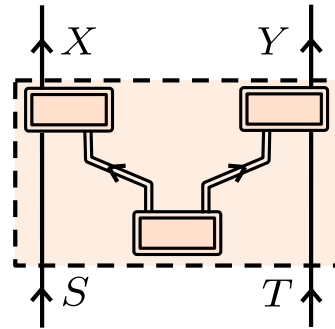
Enveloping theory: quantum/GPT processes compatible with this causal structure

Free subtheory classical processes compatible with this causal structure

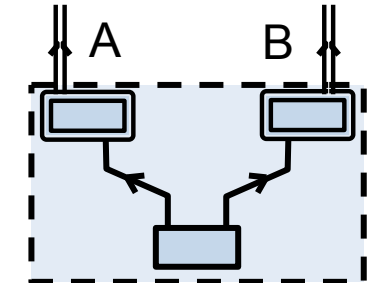
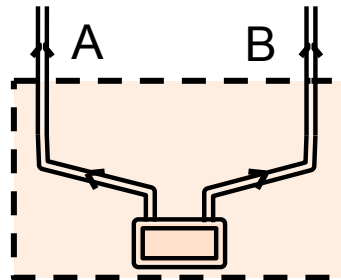
Types of resources within a 2-party network with a common-cause structure

Box-type Resources:

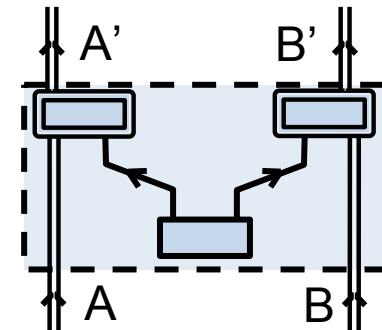
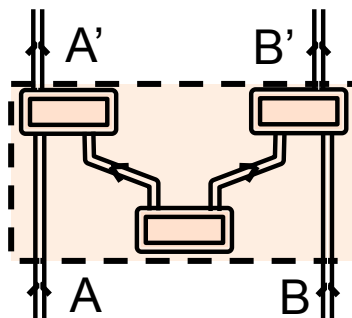
Classical inputs
and outputs



State Resources:
Quantum/GPT outputs



Channel Resources:
Quantum/GPT inputs
and outputs

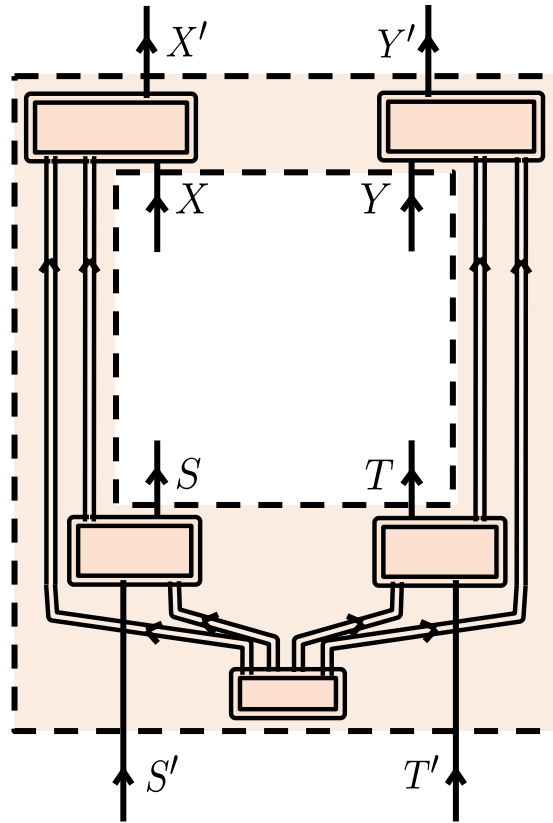


Enveloping theory

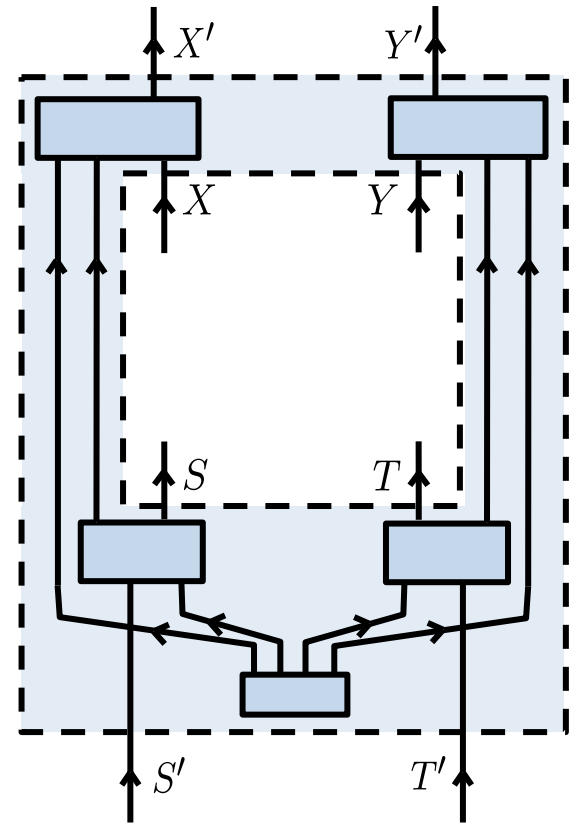
Free subtheory

Types of resources within a 2-party network with a common-cause structure

Box-to-box
processing
Resources:

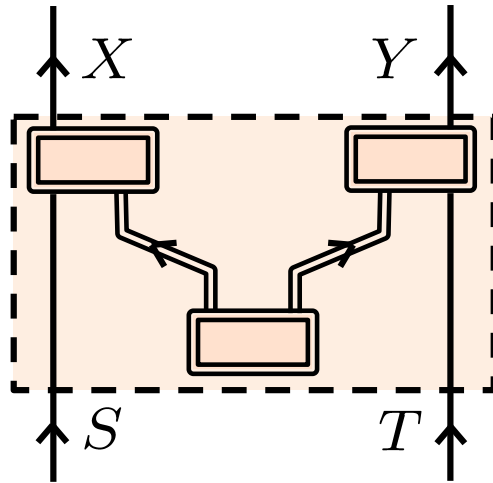


Enveloping theory

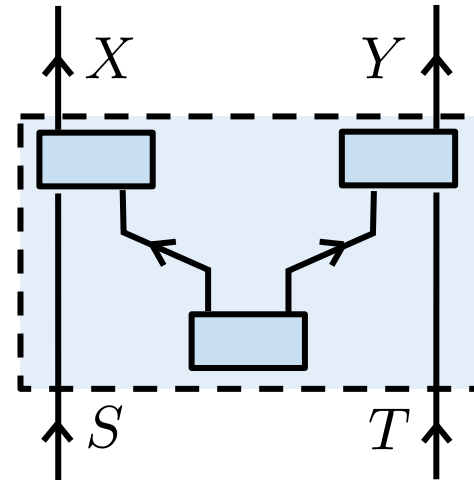


Free subtheory

We focus here on boxes

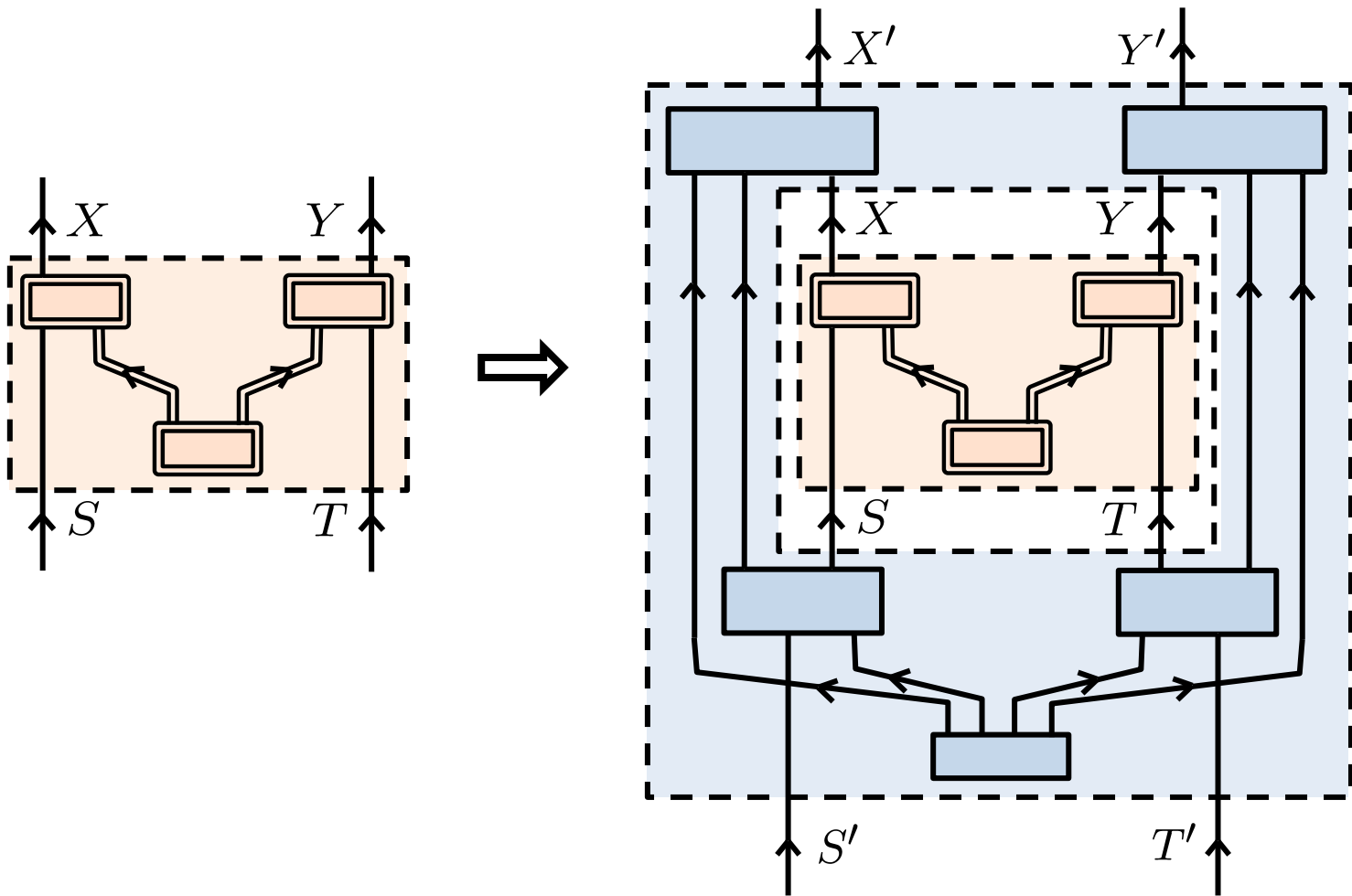


Nonclassical boxes



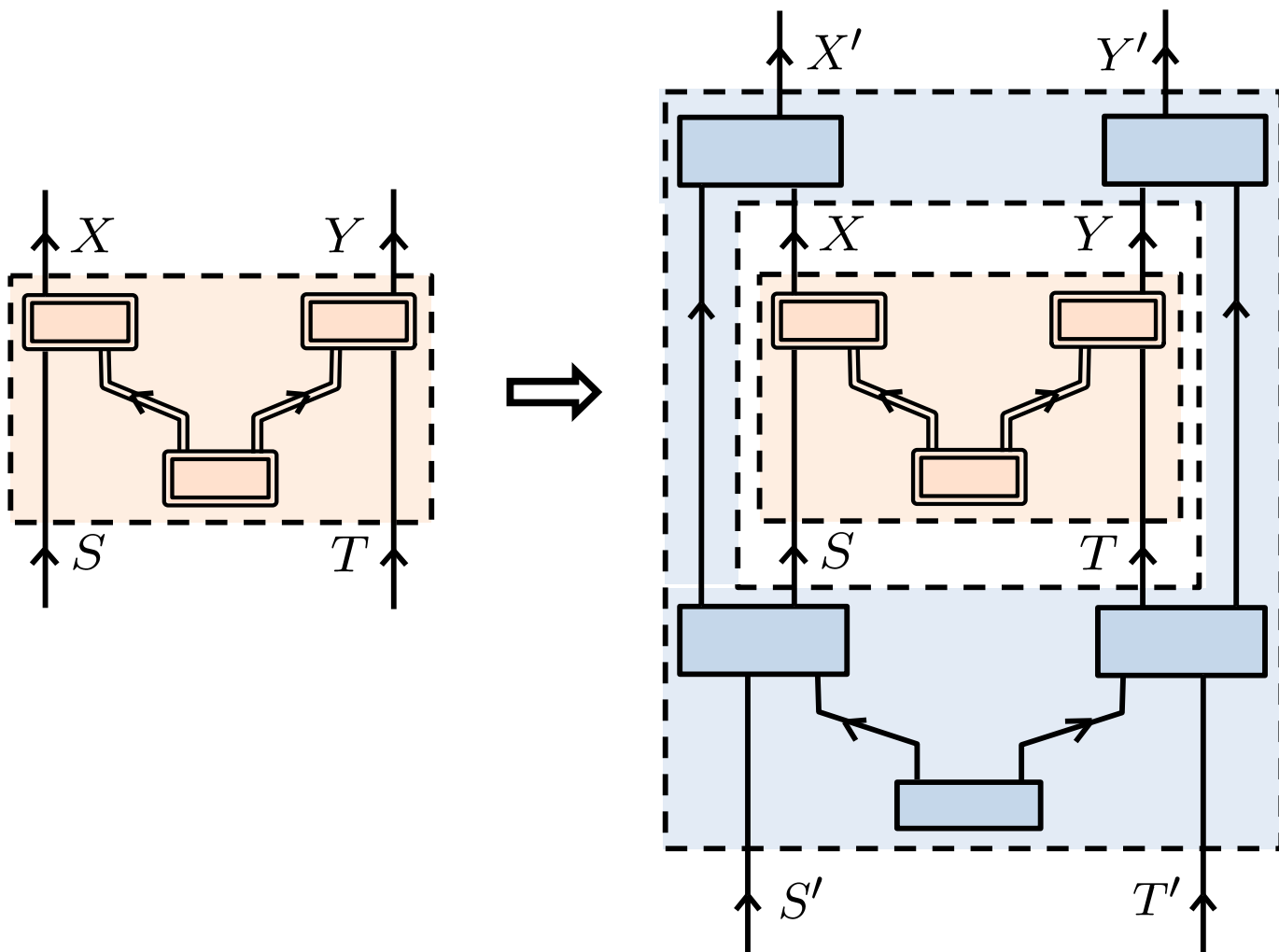
Classical boxes

Free operations taking a box to a box



$$P_{XY|ST} \mapsto P_{X'Y'|S'T'}$$

Free operations taking a box to a box



$$P_{XY|ST} \mapsto P_{X'Y'|S'T'}$$

Our definition of the free operations for 2-party networks with a common cause structure

→ Local Operations and Shared Randomness (LOSR)

As defined in:

de Vicente, *On nonlocality as a resource theory and nonlocality measures*, J. Phys. A 47, 424017 (2014)

J. Geller and M. Piani, *Quantifying non-classical and beyond-quantum correlations in the unified operator formalism*, J. Phys. A 47, 424030 (2014)

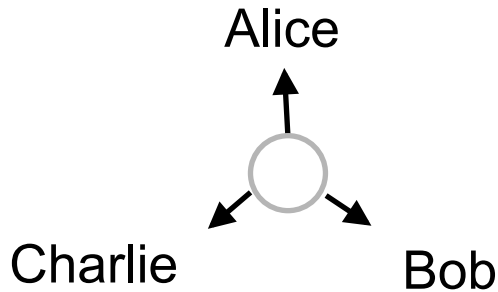
These articles motivated LOSR as the right choice using a different perspective.

Our definition of the free operations for 3-party networks with various different causal structures:

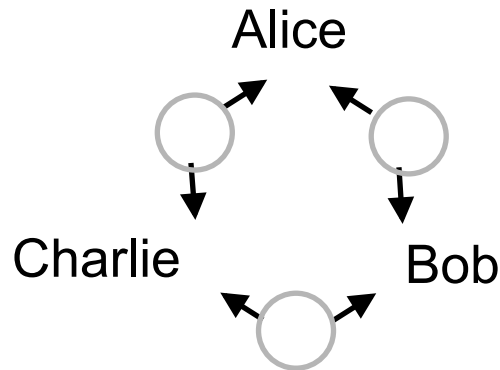
Local Operations and Causally Admissible Shared Randomness (LOCASR)

3-party networks

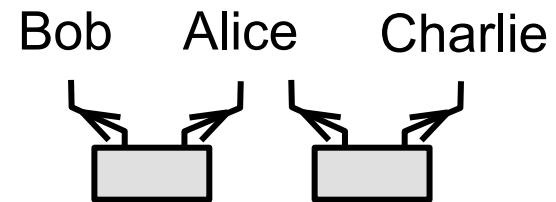
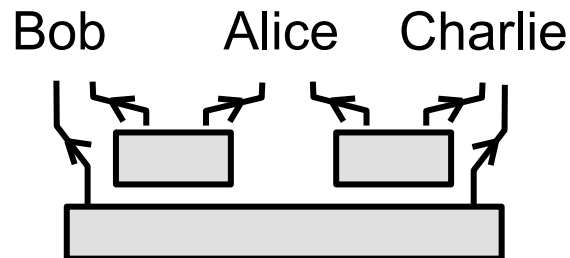
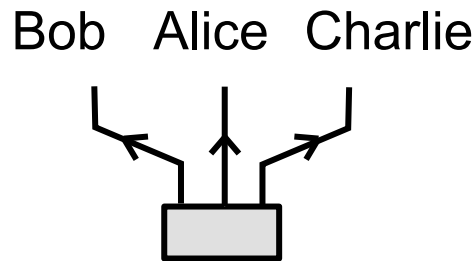
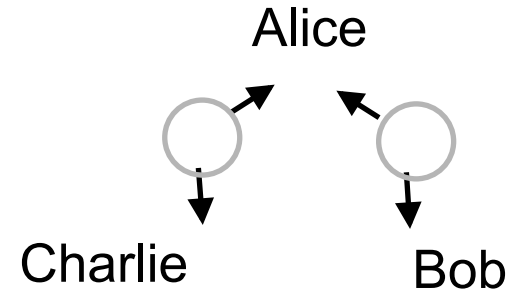
3-way common cause



Triangle pattern of common causes



Bilocality pattern of common causes

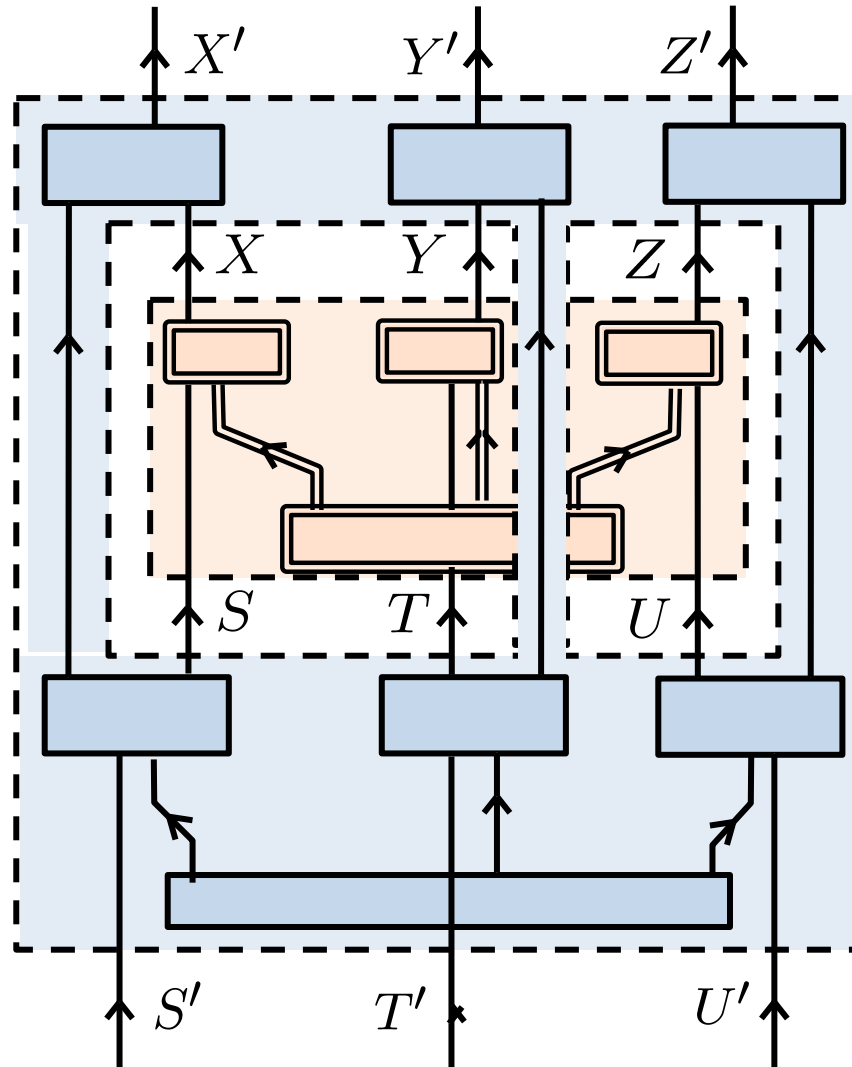
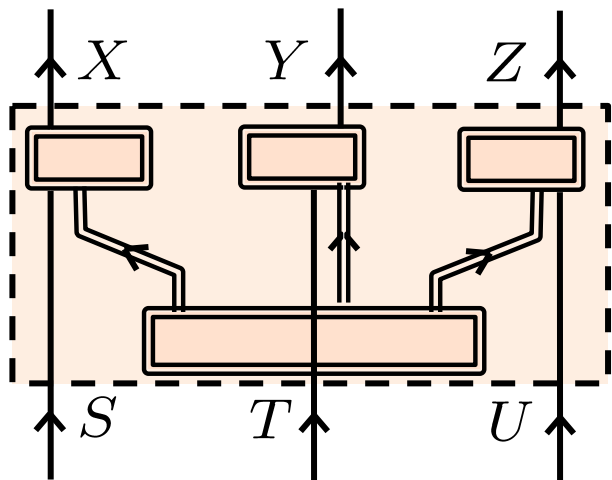


Enveloping theory: quantum/GPT processes compatible with this causal structure

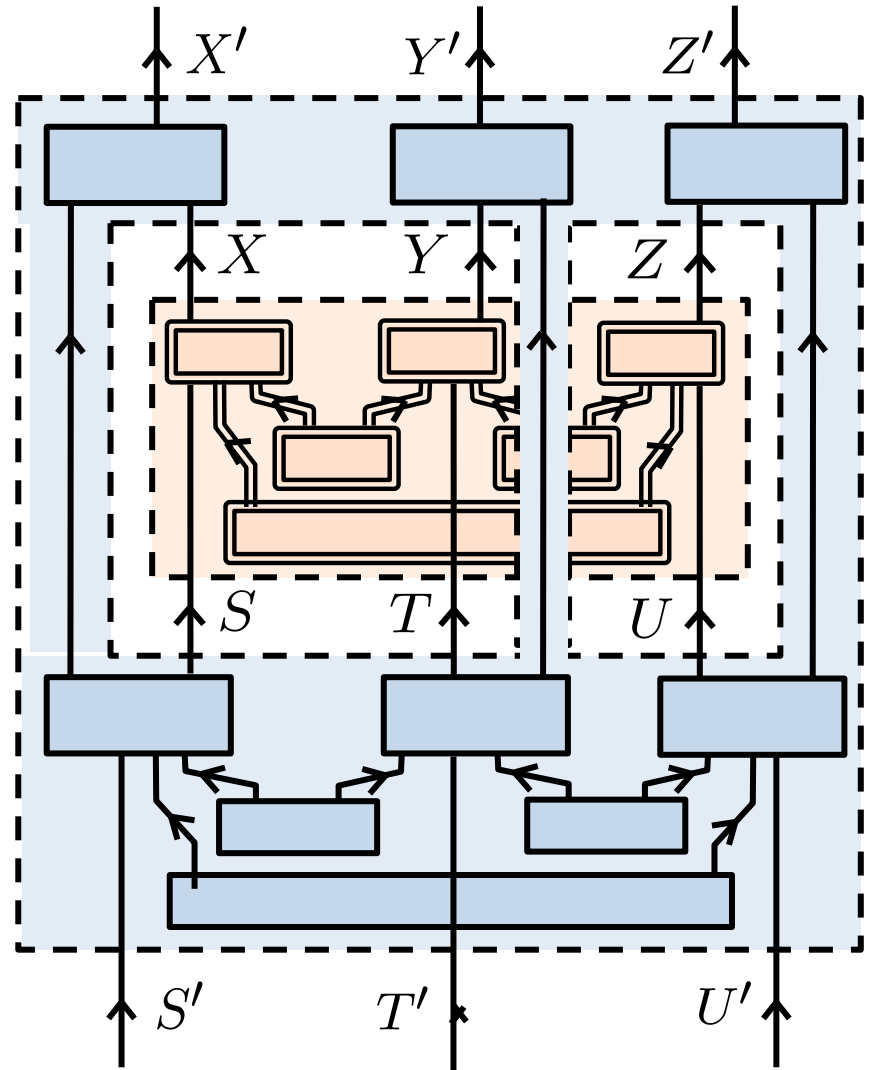
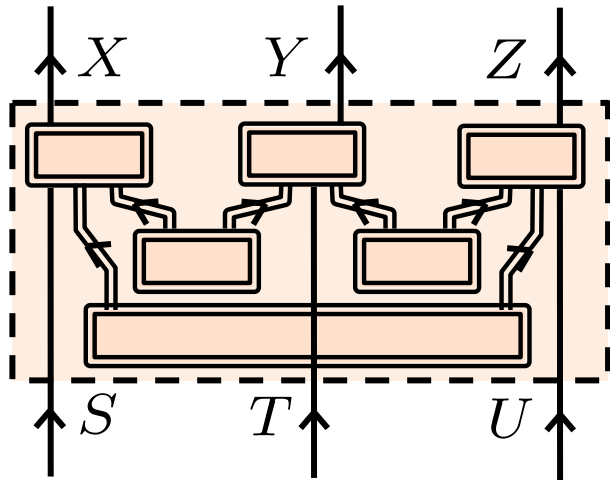
Free subtheory

classical processes compatible with this causal structure

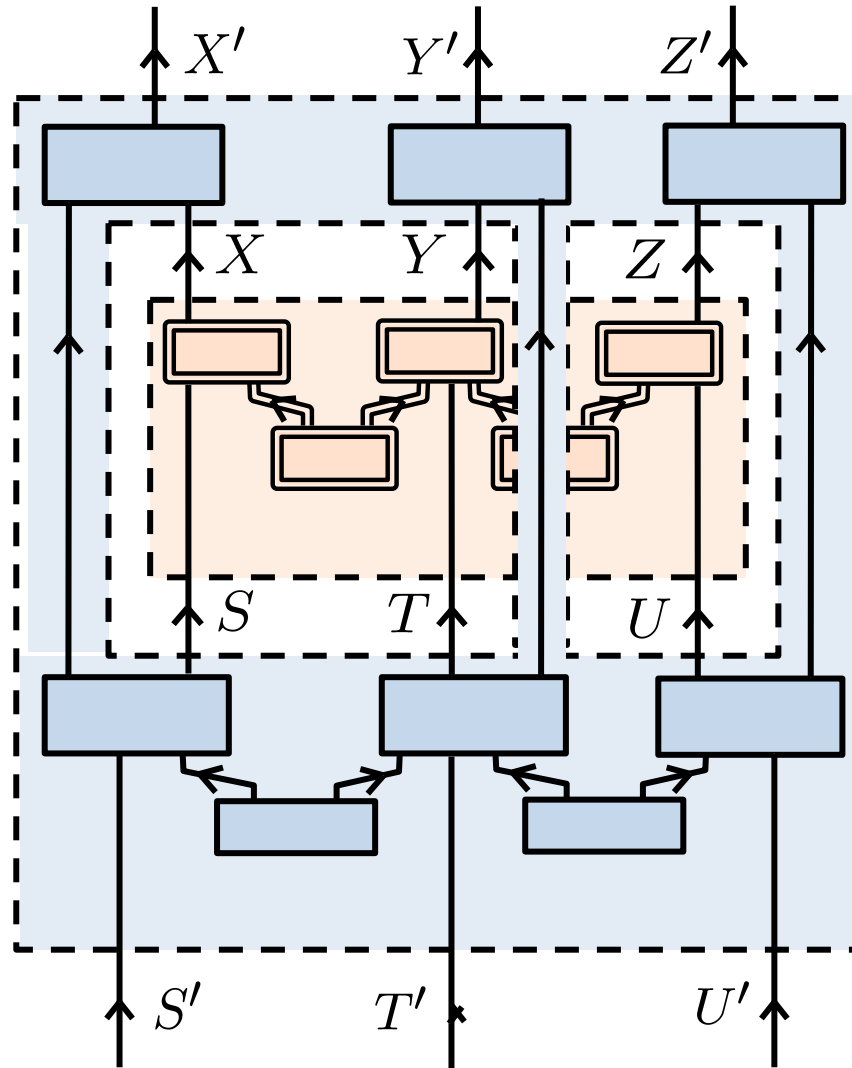
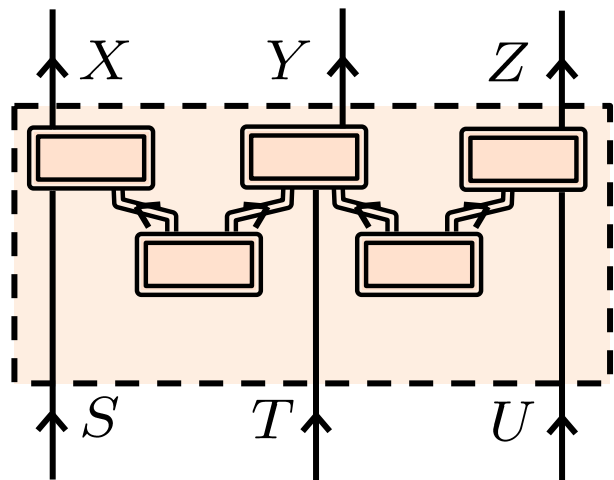
Free operations on boxes for 3-way common cause



Free operations on boxes for triangle pattern of common causes



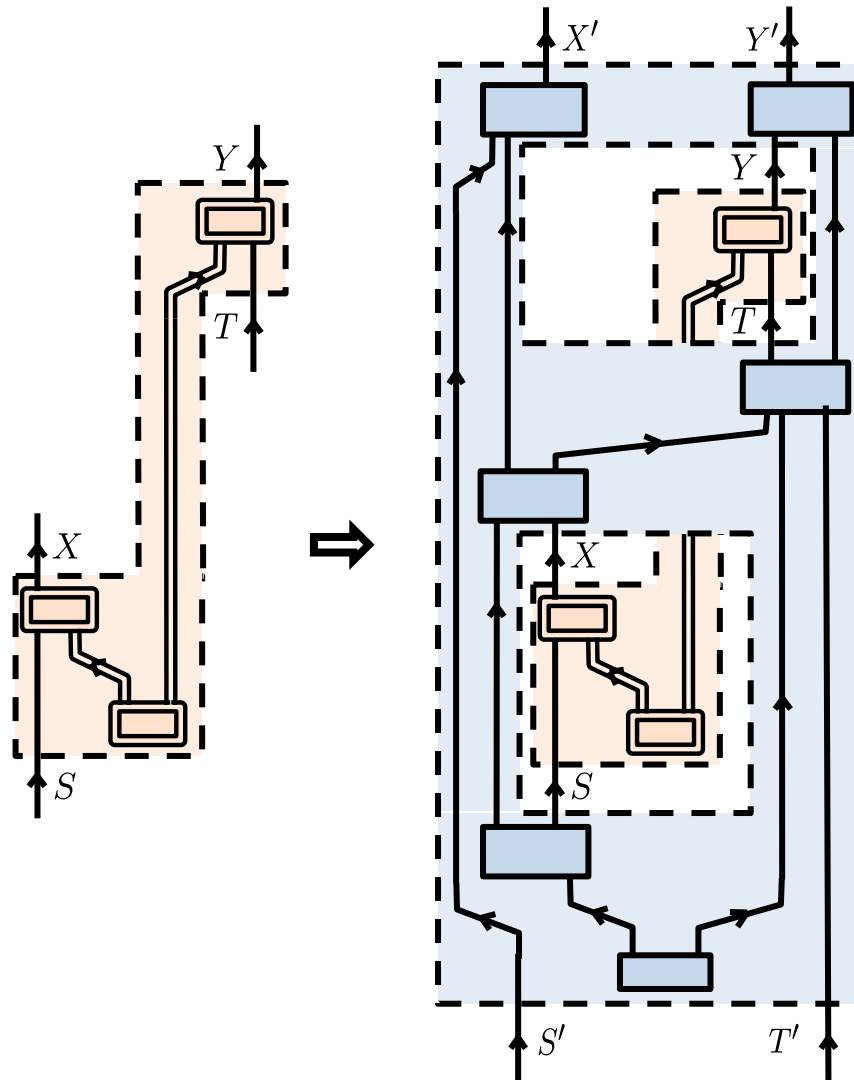
Free operations on boxes for bilocality pattern of common causes



Benefits of the causal perspective on
defining the free operations
even in the case of the 2-party network

Wirings and Prior-to-input Classical Communication (WPICCC)

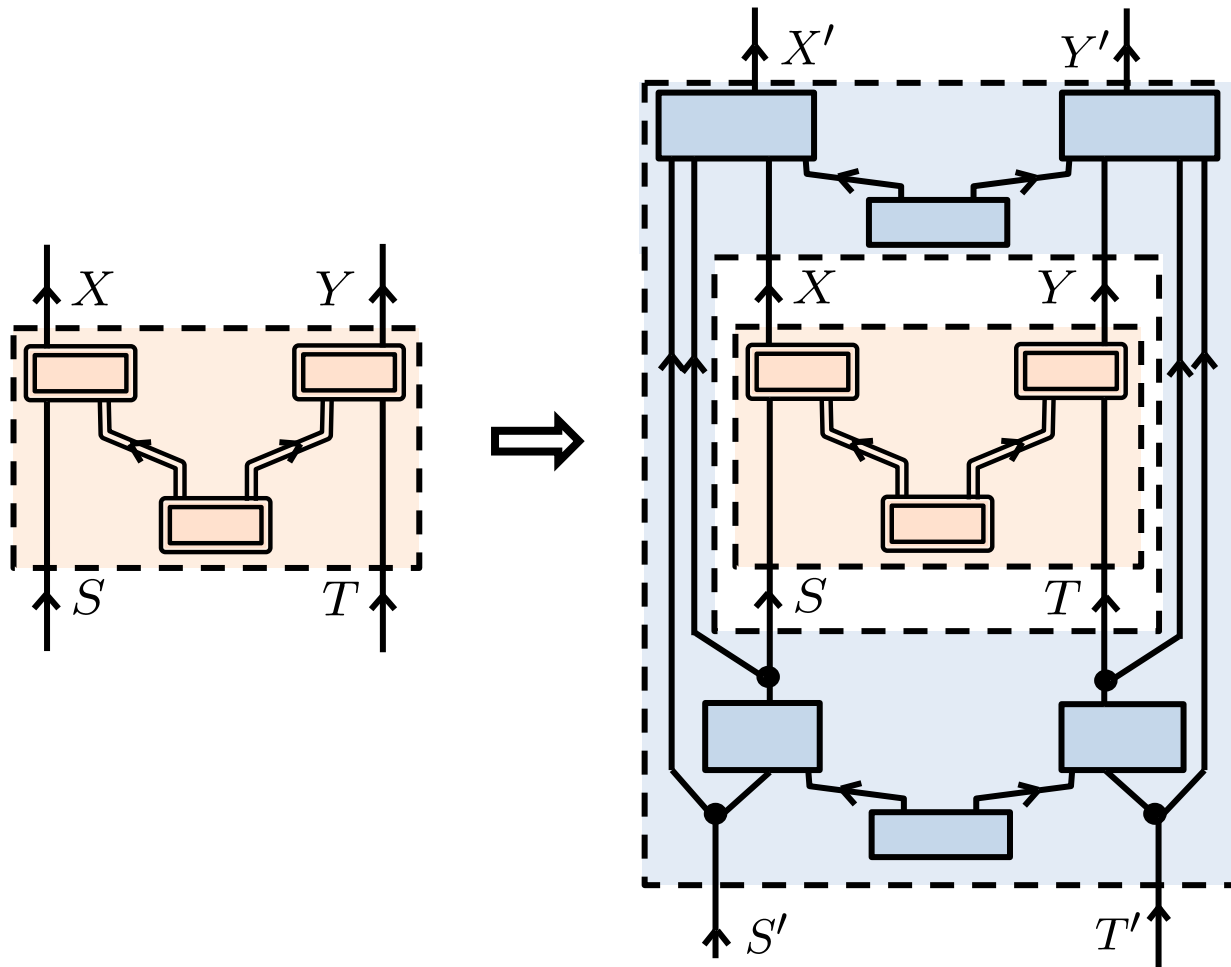
Gallego, Wurfli, Acin, Navascués, PRL 109, 070401 (2012)



Not a good choice in our view because it does not respect the causal structure

Oversight in a previous definition of “LOSR”

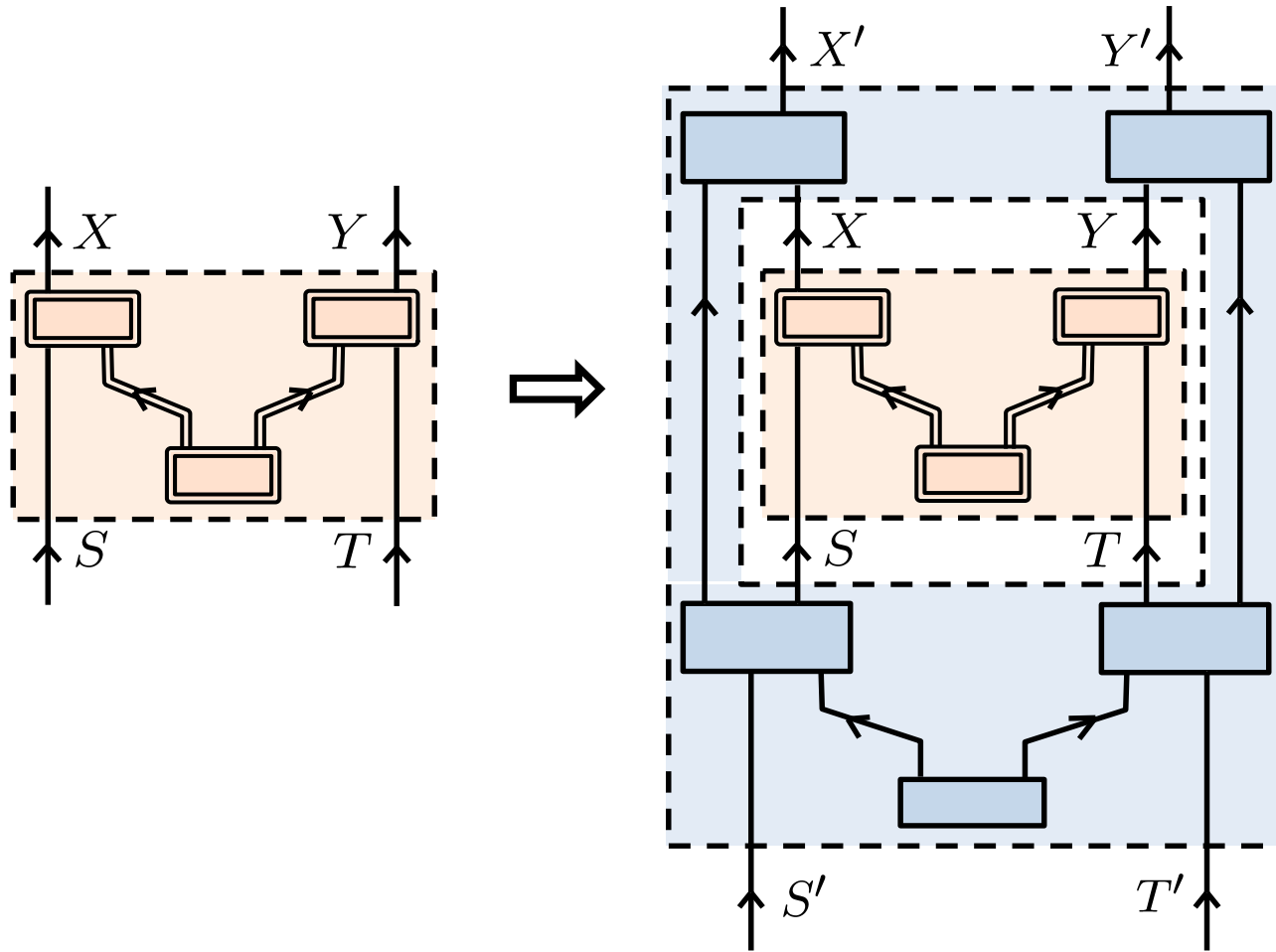
Gallego and Aolita, PRA 95 (2017)



These form a
nonconvex
strict subset
of LOSR

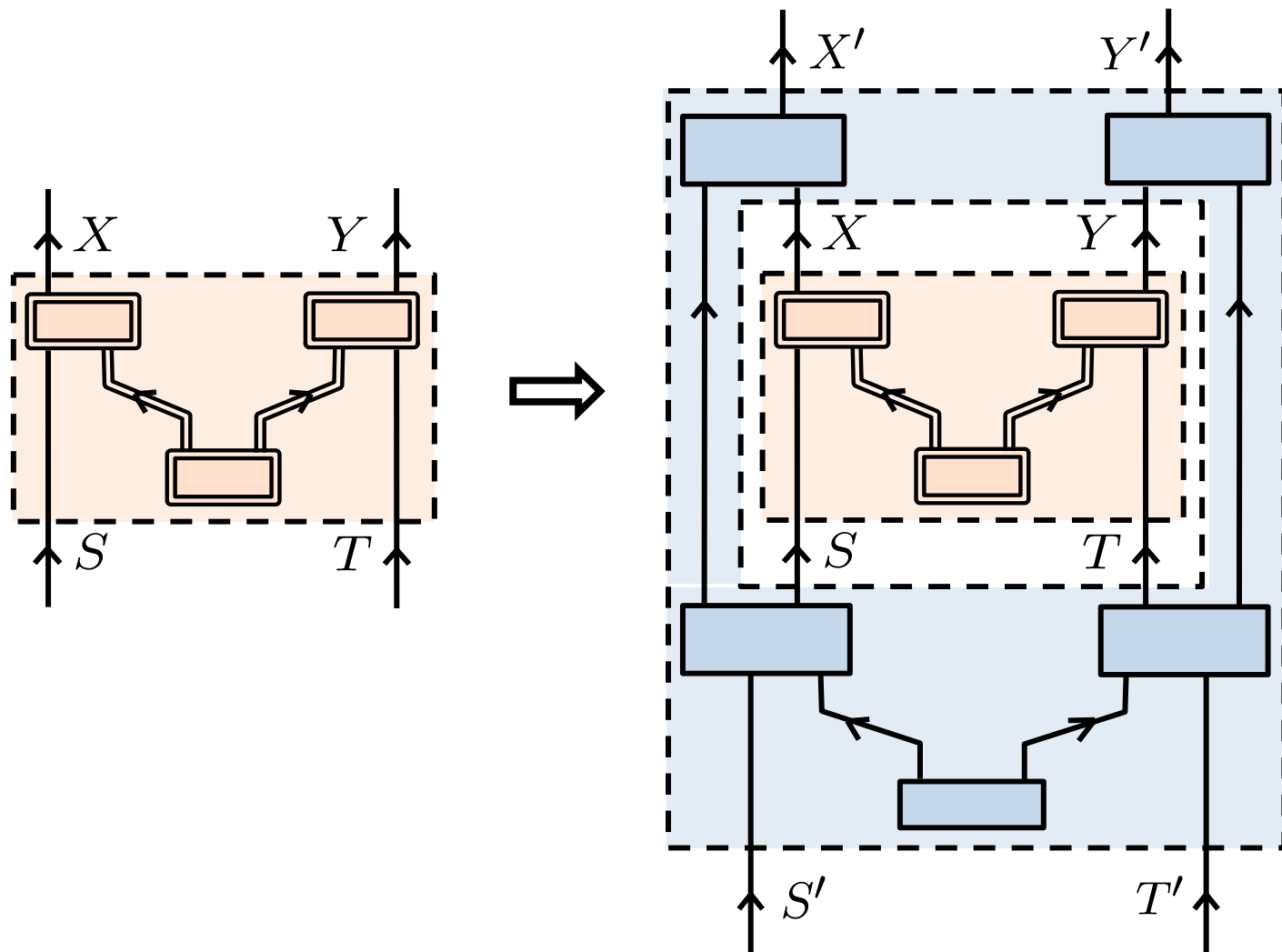
Convexification must be justified by the causal structure

Locally operations and shared randomness (LOSR)

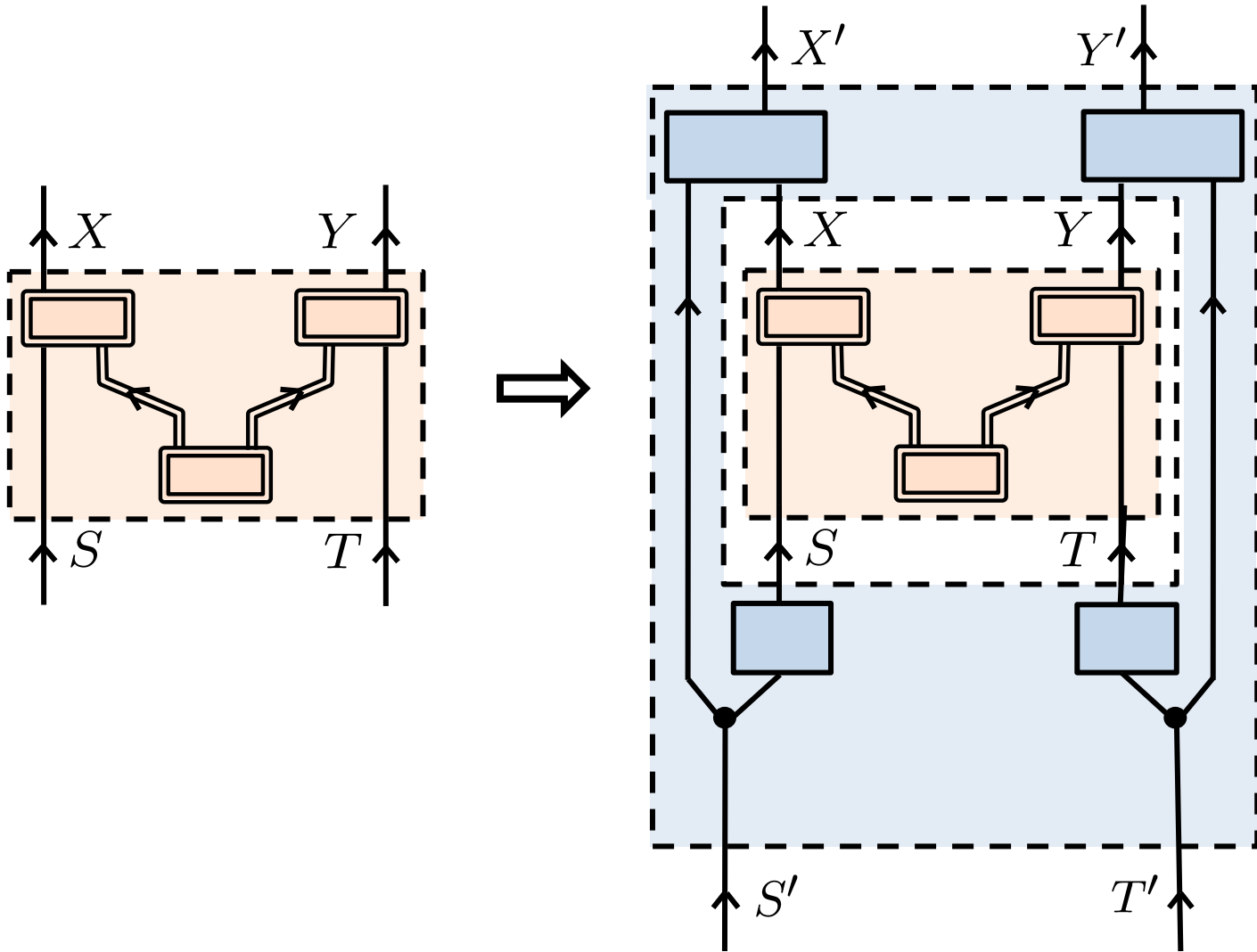


Results

Locally operations and shared randomness (LOSR)



Locally deterministic operations (LDO)



The Free Operations form a Polytope

$$\text{LOS}_{[R_1] \rightarrow [R_2]} = \text{ConvexHull} \left(\text{LDO}_{[R_1] \rightarrow [R_2]} \right)$$

Therefore

$$R_1 \mapsto R_2$$

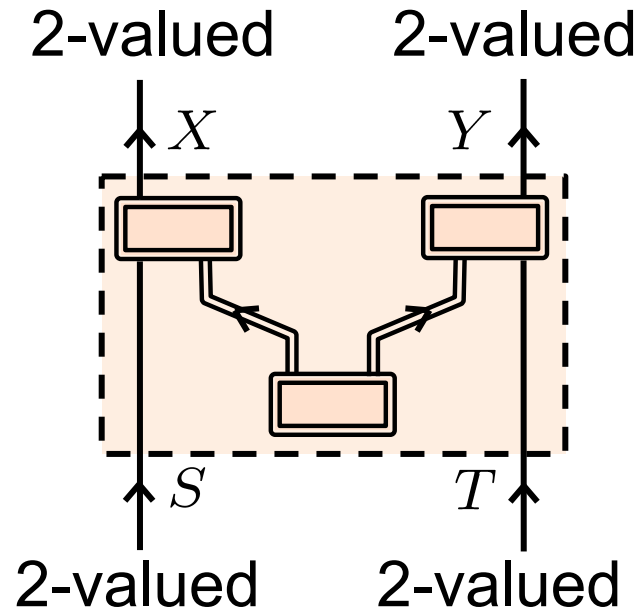
If and only if

$$R_2 \in \text{ConvexHull} \left(\text{Images}_{[R_2]}^{\text{LDO}}(R_1) \right)$$

Hence, a linear program decides convertibility

Resource monotones for the nonclassicality of common-cause boxes

Boxes of type- $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$



$\mathcal{S}_{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}}^G :=$ set of boxes of type- $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$
compatible with GPT causal model

The role of facet-defining Bell functionals

$$\begin{aligned}\text{Let } \langle A_s \rangle &:= \sum_{x \in \{0,1\}} (-1)^x P_{X|S}(x|s) \\ &= P_{X|S}(0|s) - P_{X|S}(1|s)\end{aligned}$$

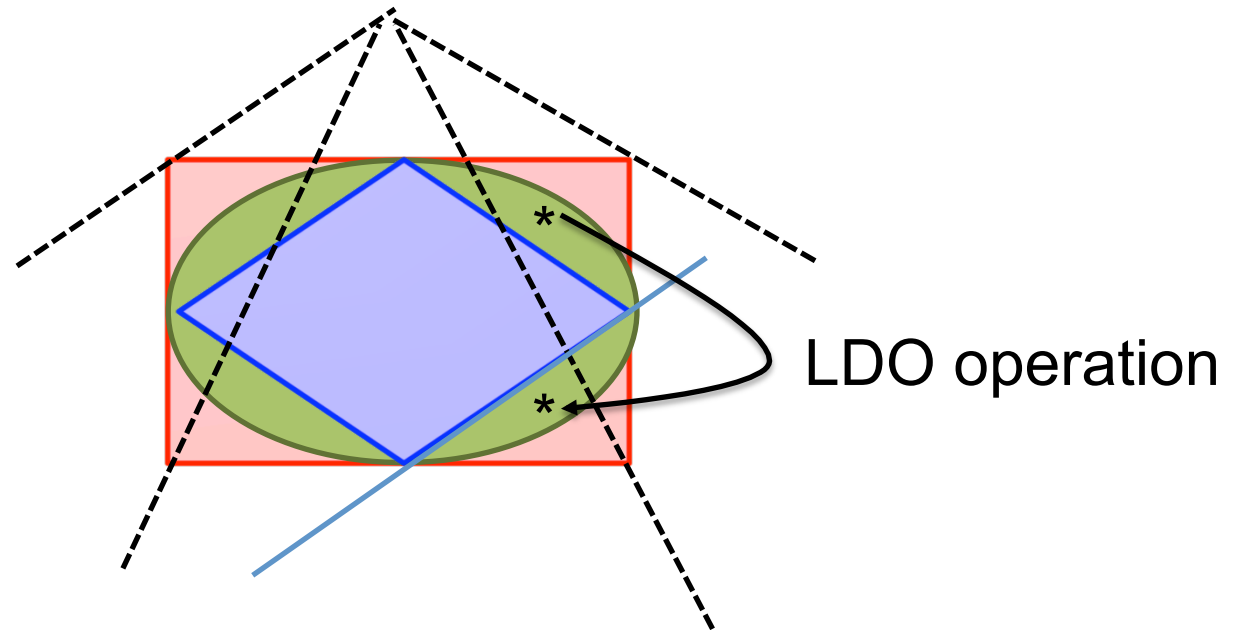
$$\begin{aligned}\langle B_t \rangle &:= \sum_{y \in \{0,1\}} (-1)^y P_{Y|T}(y|t) \\ &= P_{Y|T}(0|t) - P_{Y|T}(1|t)\end{aligned}$$

$$\langle A_t B_s \rangle := \sum_{x,y \in \{0,1\}} (-1)^{(x \oplus y)} P_{XY|ST}(xy|st).$$

$$\text{CHSH}(R) := \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

The role of facet-defining Bell functionals

They are not monotones



M_{CHSH} : monotone based on yield construction

$$\begin{aligned} M_{\text{CHSH}}(R) &:= M[\text{CHSH-yield}, \mathcal{S}_{\left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)}^G](R) \\ &= \max_{R^* \in \mathcal{S}_{\left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)}^G} \{ \text{CHSH}(R^*) \text{ s.t. } R \mapsto R^* \} \end{aligned}$$

Note: R need not be a box of type- $\left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$

M_{NPR} : monotone based on cost construction

$$\begin{aligned} M_{\text{NPR}}(R) &:= M[\text{CHSH-cost}, \mathbf{C}_{\text{NPR}}](R) \\ &= \min_{R^* \in \mathbf{C}_{\text{NPR}}} \{ \text{CHSH}(R^*) \mid R^* \mapsto R \} \end{aligned}$$

$$C_{\text{NPR}} := \{C(\alpha) : \alpha \in [0, 1]\},$$

$$C(\alpha) := \alpha R_{\text{PR}} + (1-\alpha)L_{\text{NPR}}^{\text{b}}$$

$$L_{\text{NPR}}^{\text{b}} = \frac{1}{2}R_{\text{PR}} + \frac{1}{2}L_{\emptyset}$$

[illegible]

Closed-form expression for M_{CHSH} for boxes of type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$$R \in \mathbf{S}_{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}}^{\text{free}} \quad M_{\text{CHSH}}(R) = 2$$

$$R \in \mathbf{S}_{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}}^G \quad M_{\text{CHSH}}(R) = \text{CHSH}_k(R)$$

for the unique $k \in \{0, \dots, 7\}$ s.t.

$$\text{CHSH}_k(R) > 2$$

The eight facet-defining CHSH functional

$$\begin{aligned}\text{CHSH}_0(R) &:= +\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle, \\ \text{CHSH}_1(R) &:= +\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle - \langle A_0 B_1 \rangle + \langle A_1 B_1 \rangle, \\ \text{CHSH}_2(R) &:= +\langle A_0 B_0 \rangle - \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_1 \rangle, \\ \text{CHSH}_3(R) &:= -\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_1 \rangle, \\ \text{CHSH}_4(R) &:= -\langle A_0 B_0 \rangle - \langle A_1 B_0 \rangle - \langle A_0 B_1 \rangle + \langle A_1 B_1 \rangle, \\ \text{CHSH}_5(R) &:= -\langle A_0 B_0 \rangle - \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle, \\ \text{CHSH}_6(R) &:= -\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle - \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle, \\ \text{CHSH}_7(R) &:= +\langle A_0 B_0 \rangle - \langle A_1 B_0 \rangle - \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle.\end{aligned}$$

The eight facet-defining CHSH inequalities

$$\text{CHSH}_k(R) \leq 2 \quad \forall k$$

Closed-form expression for M_{CHSH} for boxes of type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$$R \in \mathbf{S}_{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}}^{\text{free}} \quad M_{\text{CHSH}}(R) = 2$$

$$R \in \mathbf{S}_{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}}^G \quad M_{\text{CHSH}}(R) = \text{CHSH}_k(R)$$

for the unique $k \in \{0, \dots, 7\}$ s.t.

$$\text{CHSH}_k(R) > 2$$

Many previous proposals for measures of nonlocality can be shown to be simple functions of M_{CHSH}

Closed-form expression for M_{NPR} for boxes of type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$$R \in \mathbf{S}_{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}}^{\text{free}} \quad M_{\text{NPR}}(R) = 2$$

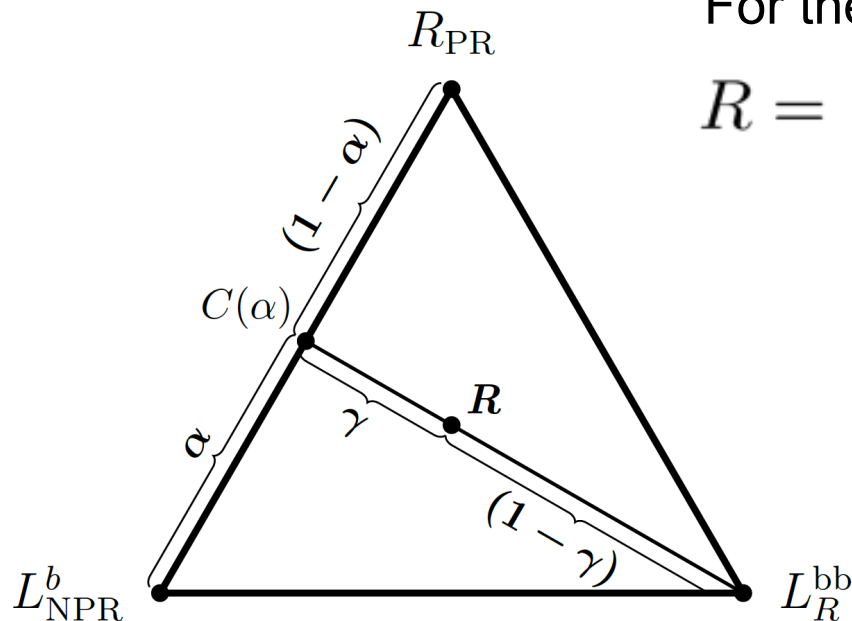
Consider the unique $k \in \{0, \dots, 7\}$ s.t. $\text{CHSH}_k(R) > 2$

$$R \in \mathbf{C}_{\text{NPR},k} \quad M_{\text{NPR}}(R) = \text{CHSH}_k(R)$$

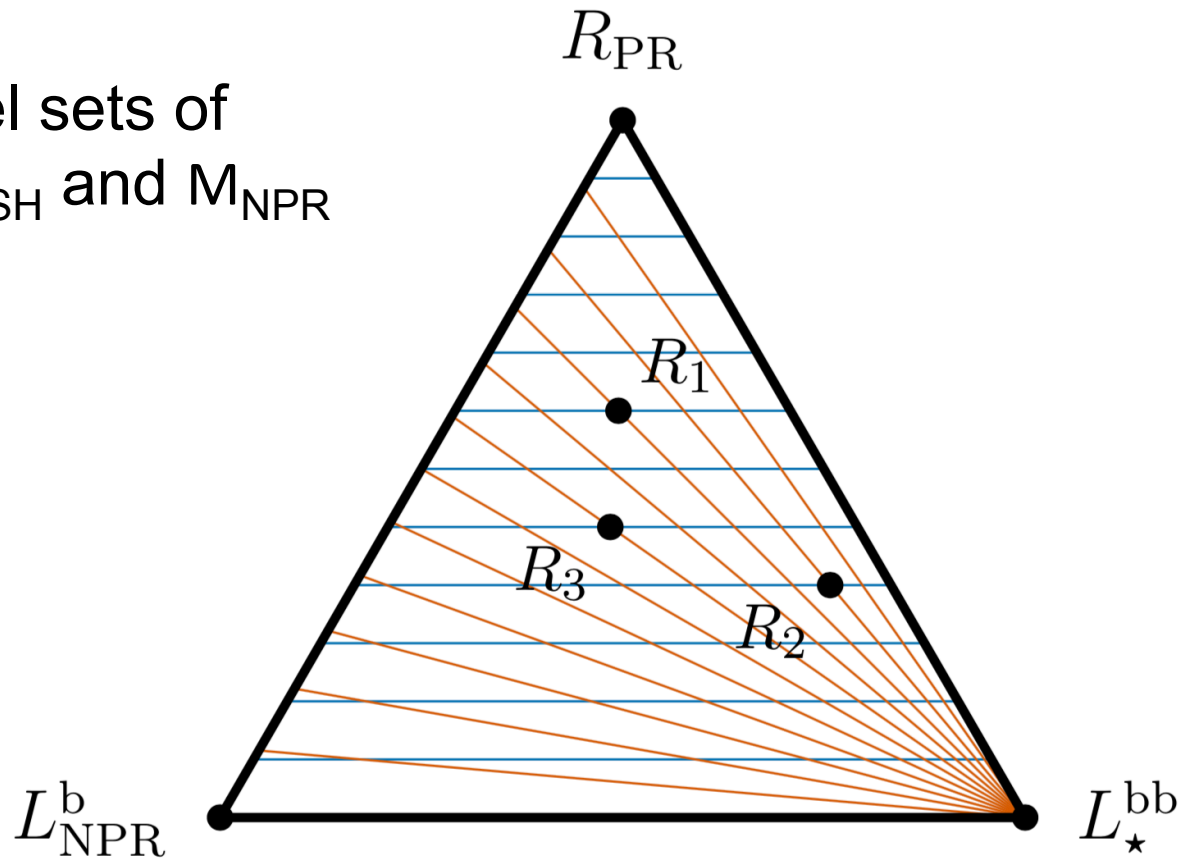
$$R \notin \mathbf{C}_{\text{NPR},k} \quad M_{\text{NPR}}(R) = 2\alpha + 2$$

For the unique decomposition

$$R = \gamma L_R^{\text{bb}} + (1-\gamma)C_k(\alpha)$$

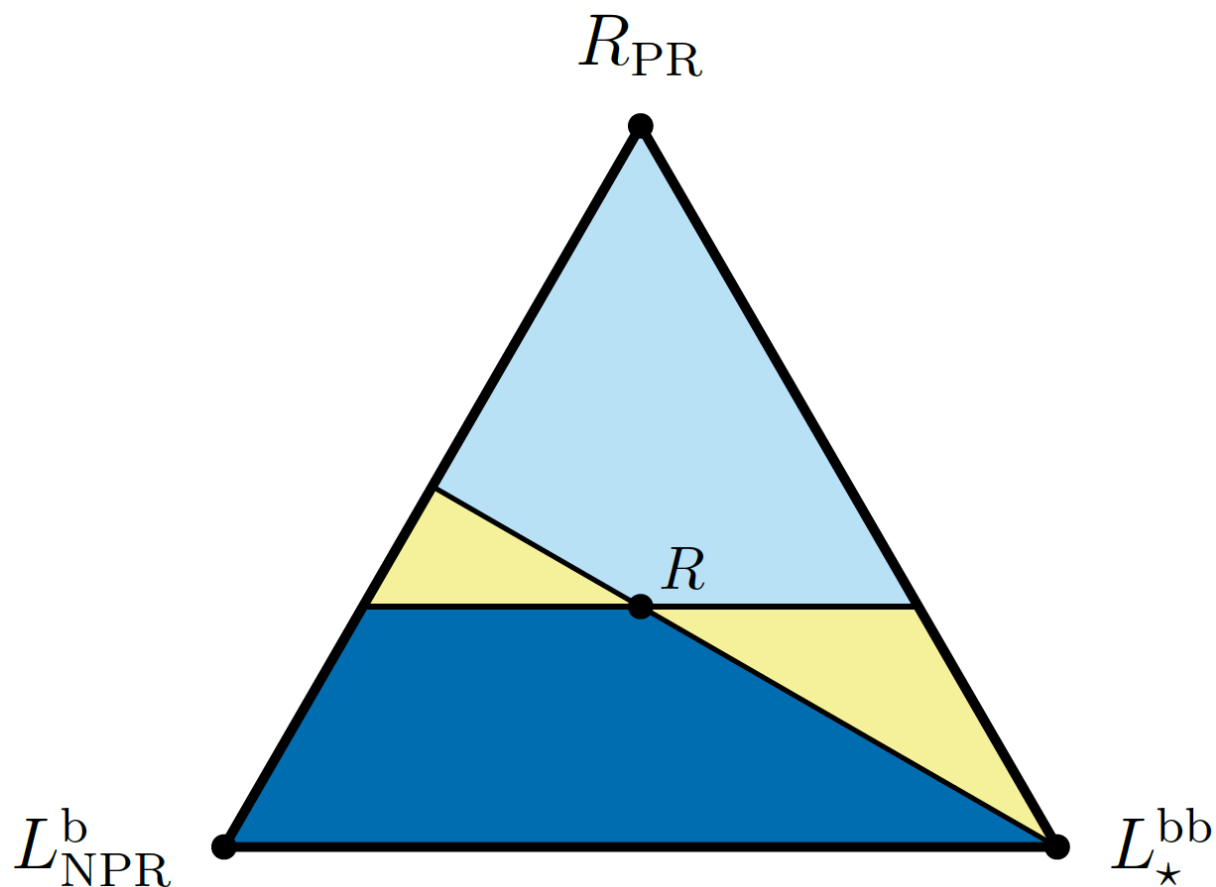


Level sets of
 M_{CHSH} and M_{NPR}



- Not a total preorder
- Not weak (incomparability relation not transitive)
- Height is infinite
- Width is infinite
- Locally infinite (continuum of resources between any two)

Completeness of the two monotones for a
2-parameter family of resources of type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$



Incompleteness of the two monotones for general resources of type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

| | $\langle A_0 \rangle$ | $\langle A_1 \rangle$ | $\langle B_0 \rangle$ | $\langle B_1 \rangle$ | $\langle A_0 B_0 \rangle$ | $\langle A_1 B_0 \rangle$ | $\langle A_0 B_1 \rangle$ | $\langle A_1 B_1 \rangle$ | M_{CHSH} | M_{NPR} |
|--|-----------------------|-----------------------|-----------------------|-----------------------|---------------------------|---------------------------|---------------------------|---------------------------|-------------------|------------------|
| L_1^{bb} | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| L_2^{bb} | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 2 | 2 |
| L_3^{bb} | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 2 |
| $C(1/2)$ | 0 | 0 | 0 | 0 | $3/4$ | $3/4$ | $3/4$ | $-3/4$ | 3 | 3 |
| $\frac{1}{2}L_1^{\text{bb}} + \frac{1}{2}C(1/2)$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $7/8$ | $7/8$ | $7/8$ | $1/8$ | $5/2$ | 3 |
| $\frac{1}{2}L_2^{\text{bb}} + \frac{1}{2}C(1/2)$ | 0 | 0 | 0 | 0 | $7/8$ | $7/8$ | $3/8$ | $-3/8$ | $5/2$ | 3 |
| $\frac{1}{2}L_3^{\text{bb}} + \frac{1}{2}C(1/2)$ | 0 | 0 | 0 | 0 | $7/8$ | $3/8$ | $7/8$ | $-3/8$ | $5/2$ | 3 |

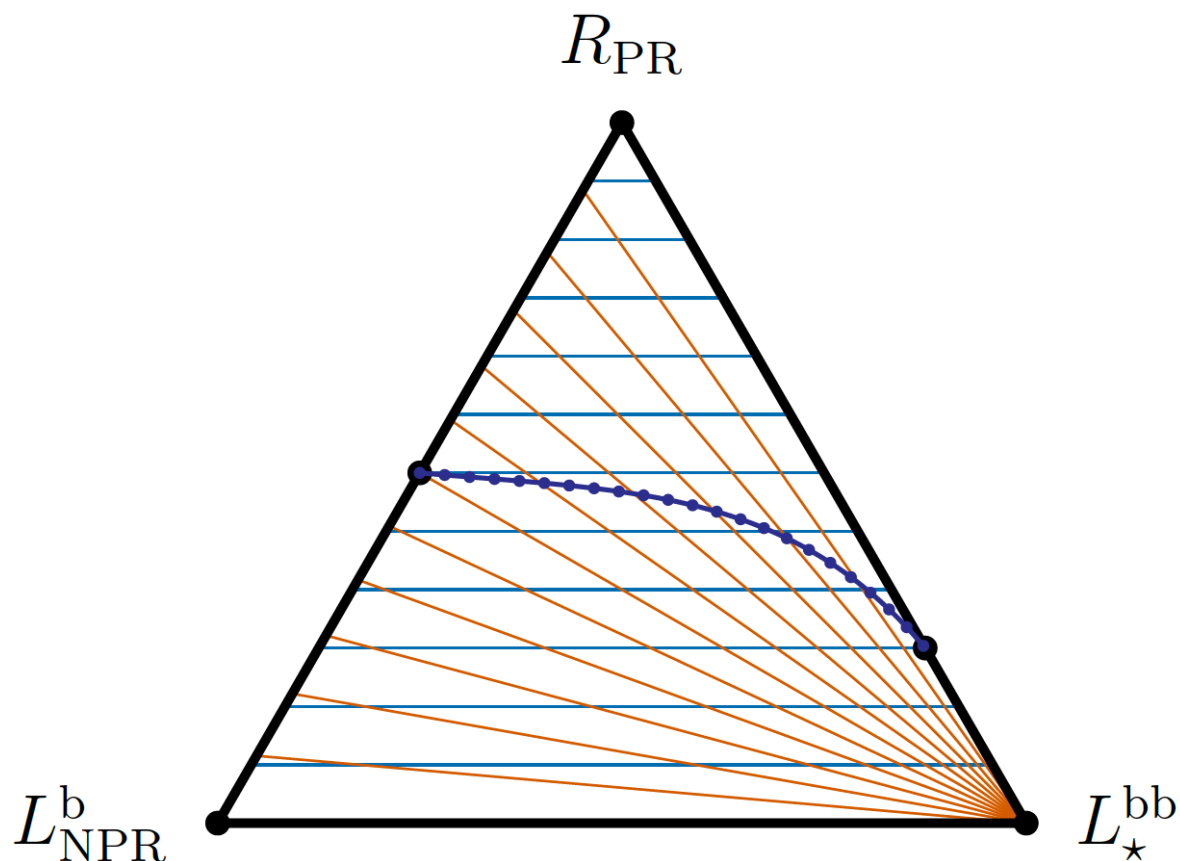
The three boxes have the same values of the 2 monotones
but

The first box is above the second one

The second and third boxes are incomparable

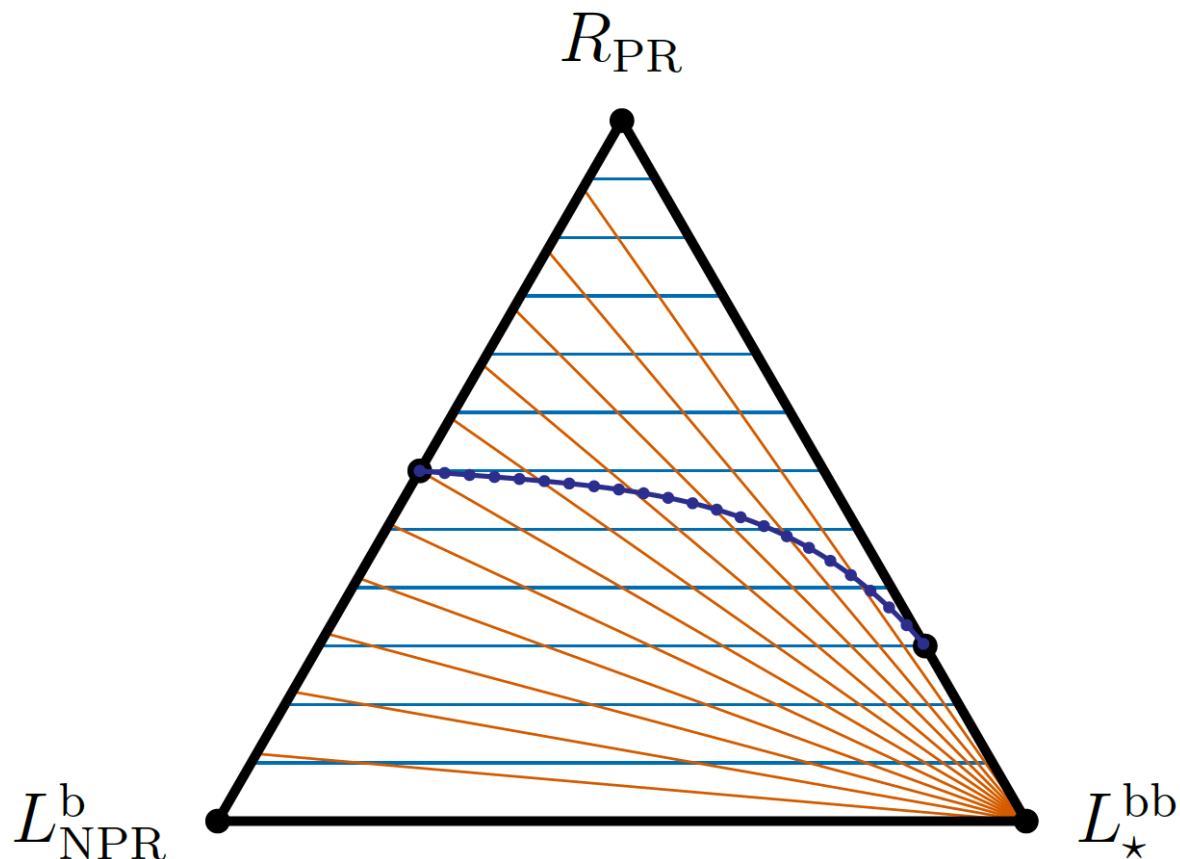
At least 8 independent monotones are needed to characterize the preorder of resources of type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

Quantumly realizable boxes



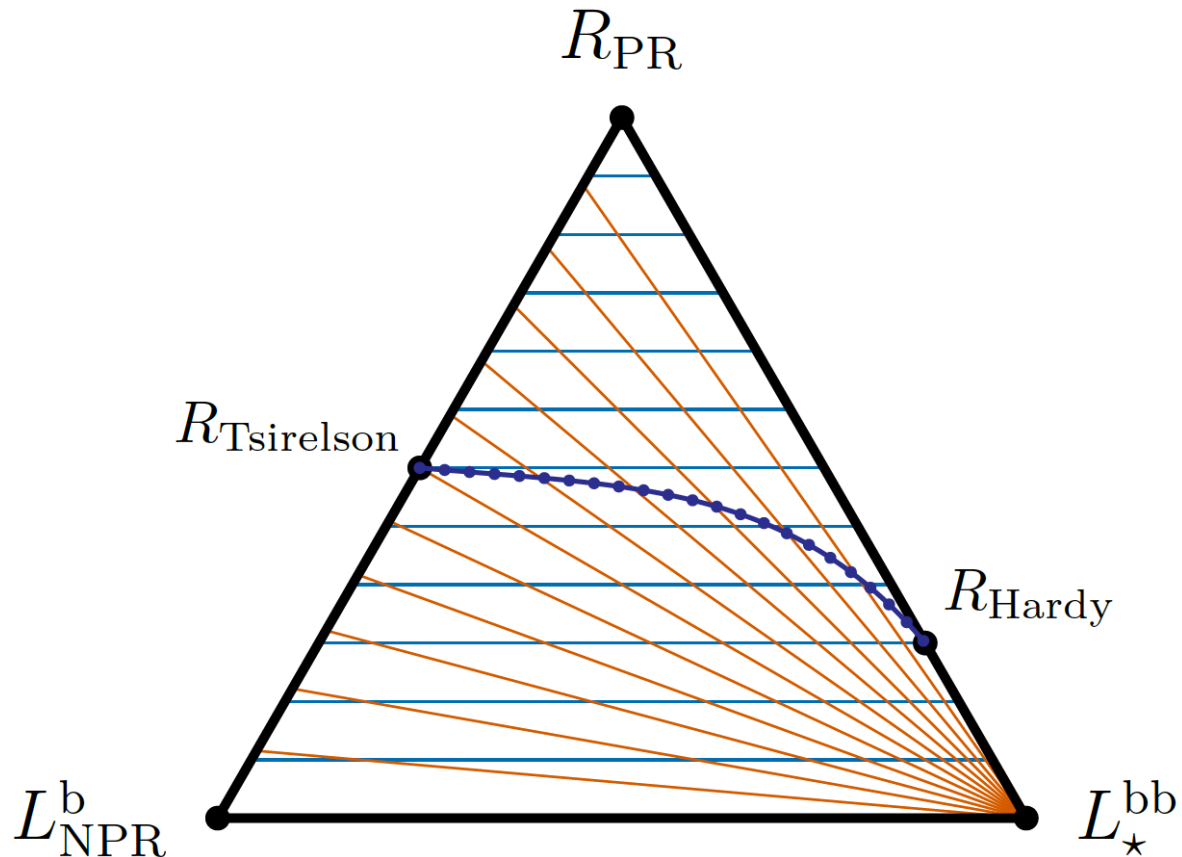
Previous conclusions about the pre-order hold here as well

Quantumly realizable boxes



No unique top of order. There are a continuum of extremal quantumly realizable boxes and these are all top-of-the-order

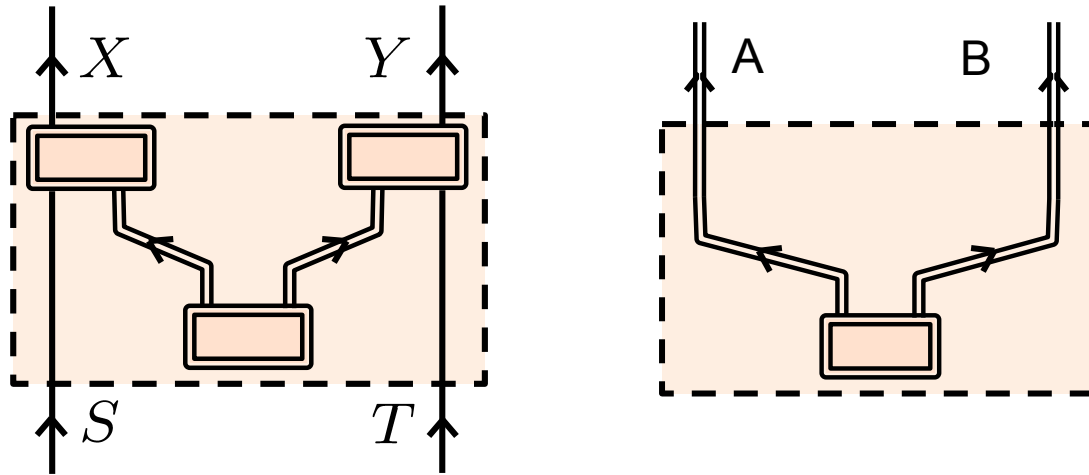
Quantumly realizable boxes



Hardy and Tsirelson boxes are each maximally nonclassical but for different measures of nonclassicality

A puzzle concerning the relation between
nonlocality and entanglement

Consider a resource theory where we allow interconversions between boxes and states



Entanglement theory says

$$|\phi^+\rangle \mapsto |\psi_{\text{Hardy}}\rangle$$

By definition

$$|\psi_{\text{Hardy}}\rangle \mapsto R_{\text{Hardy}}$$

But

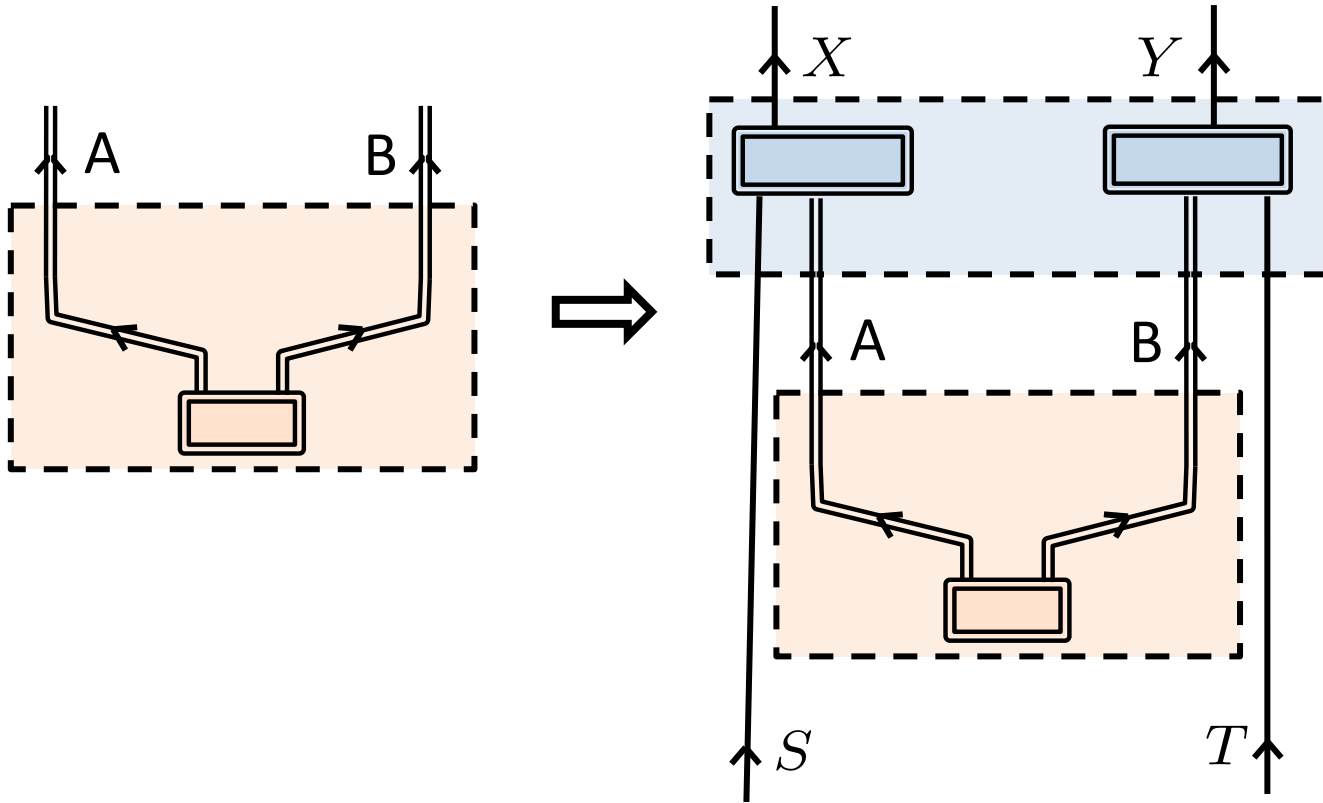
$$|\phi^+\rangle \not\mapsto R_{\text{Hardy}}$$

Apparent
inconsistency

Because in a resource theory,
conversion is a transitive relation

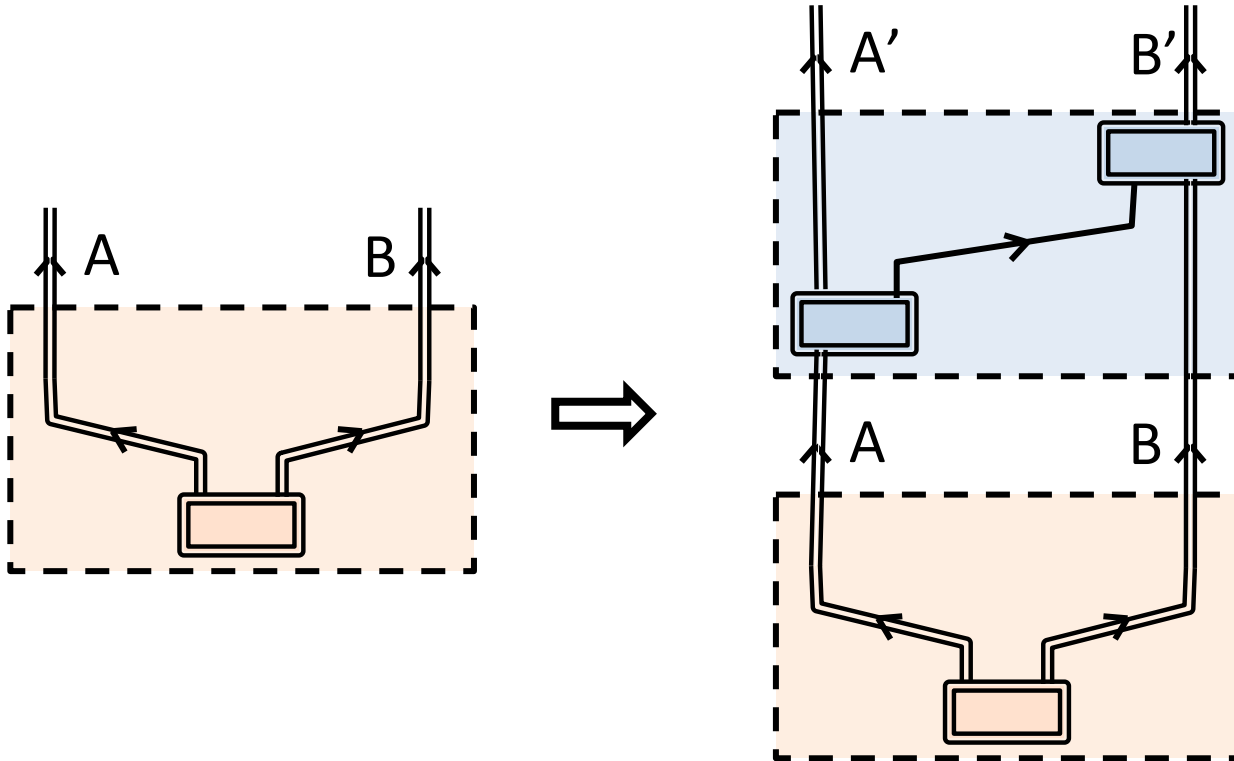
Conversion of state to box here is by LDO (hence within LOSR)

$$|\psi_{\text{Hardy}}\rangle \mapsto R_{\text{Hardy}}$$

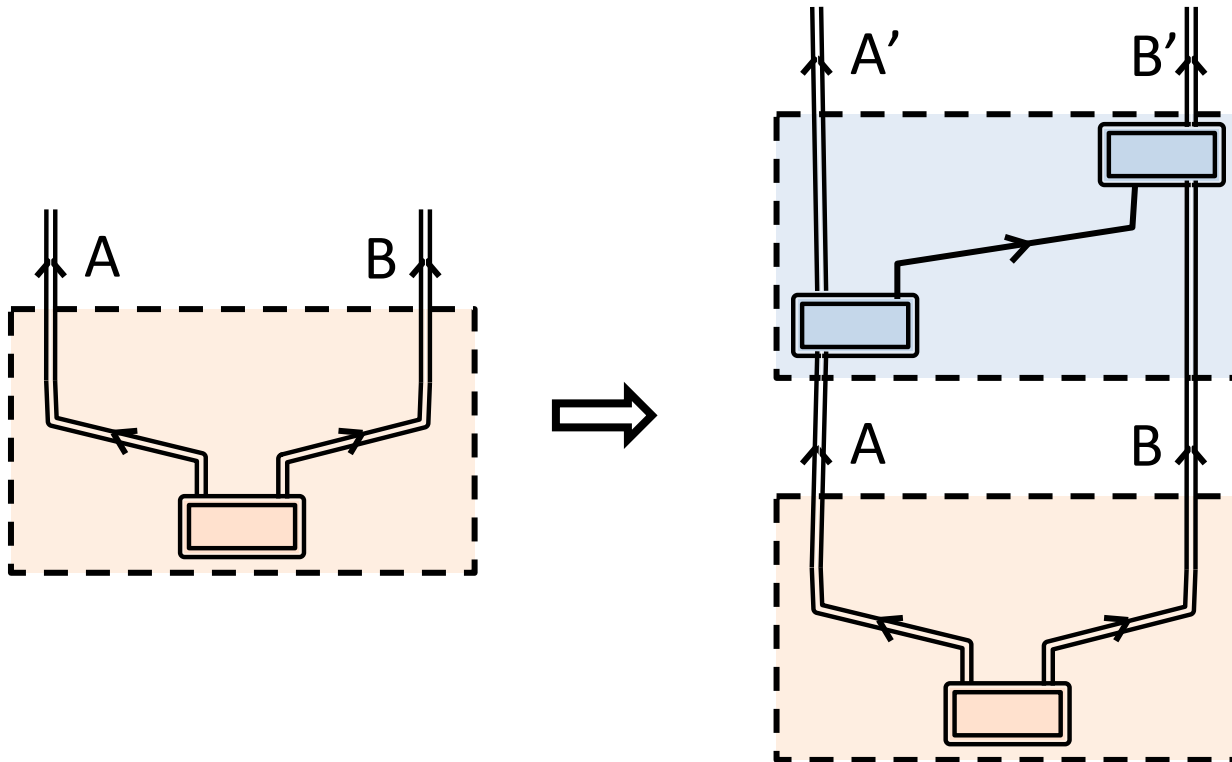


Conversion between states in entanglement theory is by **LOCC**!

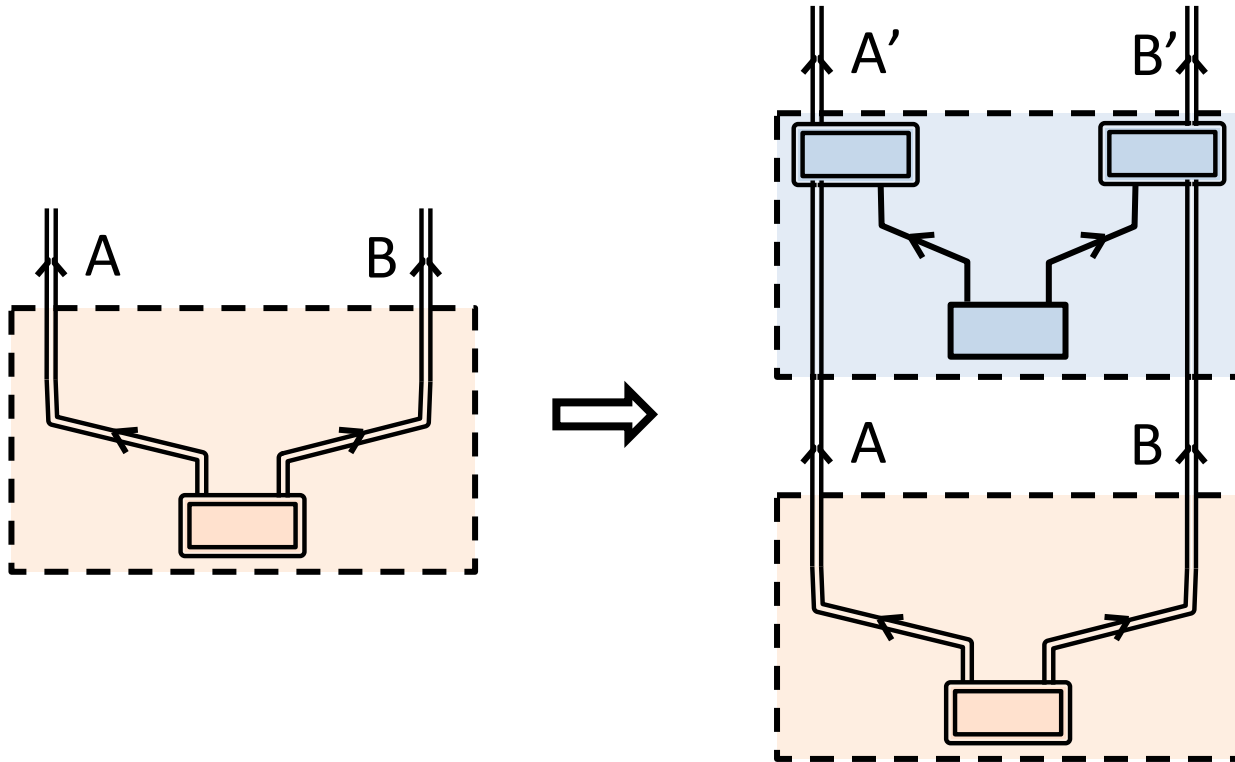
$$|\phi^+\rangle \mapsto |\psi_{\text{Hardy}}\rangle$$



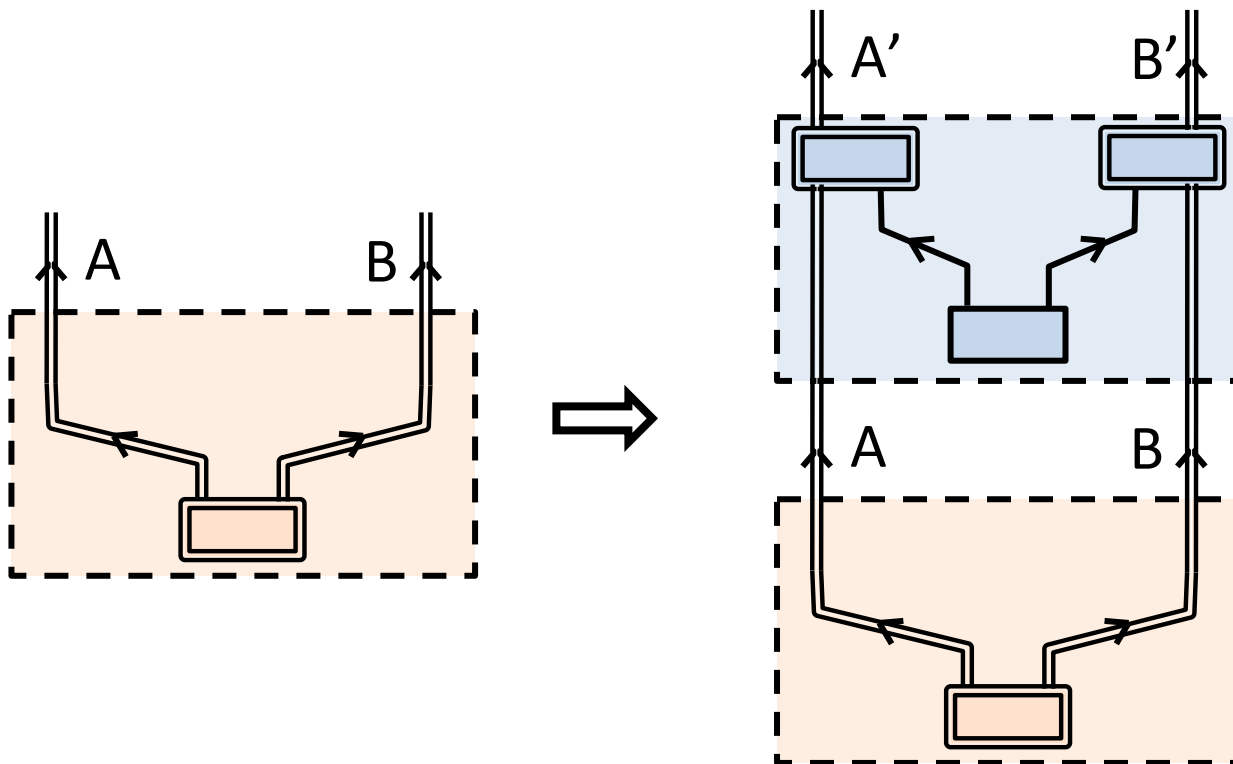
LOCC entanglement theory



LOSR entanglement theory



$$|\phi^+\rangle \not\mapsto |\psi_{\text{Hardy}}\rangle$$



Under LOCC operations

$$|\phi^+\rangle \mapsto |\psi_{\text{Hardy}}\rangle$$

Under LOSR operations

$$|\psi_{\text{Hardy}}\rangle \mapsto R_{\text{Hardy}}$$

Under LOSR operations

$$|\phi^+\rangle \not\mapsto R_{\text{Hardy}}$$

Apparent
inconsistency

Under **LOSR** operations

$$|\phi^+\rangle \not\mapsto |\psi_{\text{Hardy}}\rangle$$

Under **LOSR** operations

$$|\psi_{\text{Hardy}}\rangle \mapsto R_{\text{Hardy}}$$

Under **LOSR** operations

$$|\phi^+\rangle \not\mapsto R_{\text{Hardy}}$$

Consistent!

Relative to LOSR

$|\phi^+\rangle$ incomparable to $|\psi_{\text{Hardy}}\rangle$

just as

$R_{\text{Tsirolson}}$ incomparable to R_{Hardy}

To have a resource theory
that allows conversion between states and boxes,
one needs to develop
LOSR entanglement theory

See David Schmid's talk tomorrow

Conclusions

Taking the enveloping theory to be processes compatible with a quantum or GPT causal model, and the free subtheory to be those compatible with a classical causal model, we define a resource theory of nonclassicality for arbitrary networks

For Bell scenarios (boxes), we have determined important features of the pre-order, both for the GPT and the quantum cases

Although violations of facet-defining Bell inequalities are sufficient to witness nonclassicality, they are not sufficient to quantify it

Open problems

Determine which parameters of a box are necessary and sufficient to determine its equivalence class

Find a complete set of monotones for boxes of type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

Consider asymptotic conversion, catalysis, etcetera

Consider boxes in multipartite Bell scenarios. What new features emerge?

Consider boxes in triangle, bilocality, etcetera. What new features emerge? (note: nonconvex!)

Consider interconversion of boxes and states

Explore applications to device-independent cryptography

For more details, see:
arXiv:1903.06311

Thanks!

Online applications for Perimeter Institute postdoctoral
fellowships open on Nov. 15.
Ask me about the new Causal Inference Initiative