Quantum Causal Models

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Causality in the quantum world

Capri, September 2019

J.-M. Allen, JB, D. Horsman, C. Lee, R. W. Spekkens., PRX 7, 031021 (2017).
JB, R. Lorenz, O. Oreshkov, arXiv:1906.10726.

Introduction

The framework of classical causal models describes classical random variables with specified causal relationships between them.

The causal relationships induce constraints on probability distributions.

The framework is useful in many contexts. It enables us to make inferences about causal structure in cases where we have observed data, but the causal structure is unknown.

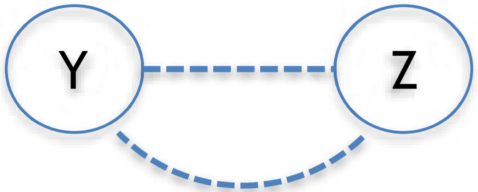
This in turn enables us to make deductions about what will happen in alternative scenarios, e.g., if I intervene and fix a variable to have the value that I want, what happens to the other variables? It also enables a rigorous account of counterfactual statements.

The main aim of this talk is to describe a framework for quantum causal models.

Assume finitely-valued random variables and finite dimensional Hilbert spaces throughout!

The classical notion of common cause

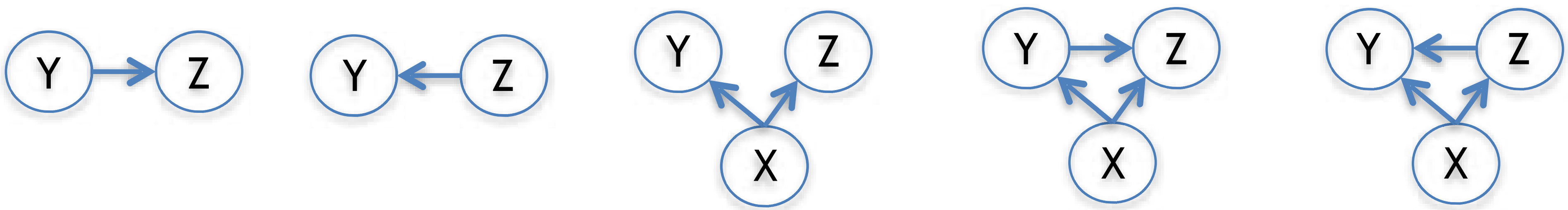
Reichenbach's principle



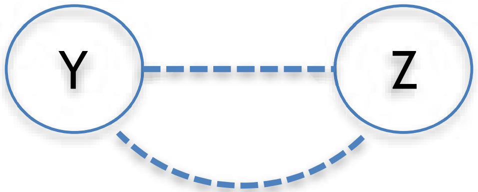
Y, Z are classical random variables.

Suppose they are *correlated*, i.e., P(YZ) ≠ P(Y) P(Z).

Then: one variable is a cause of the other, or there is a common cause, or both:



Reichenbach's principle



This one's special, because in this case, the principle also implies a constraint on the probabilities

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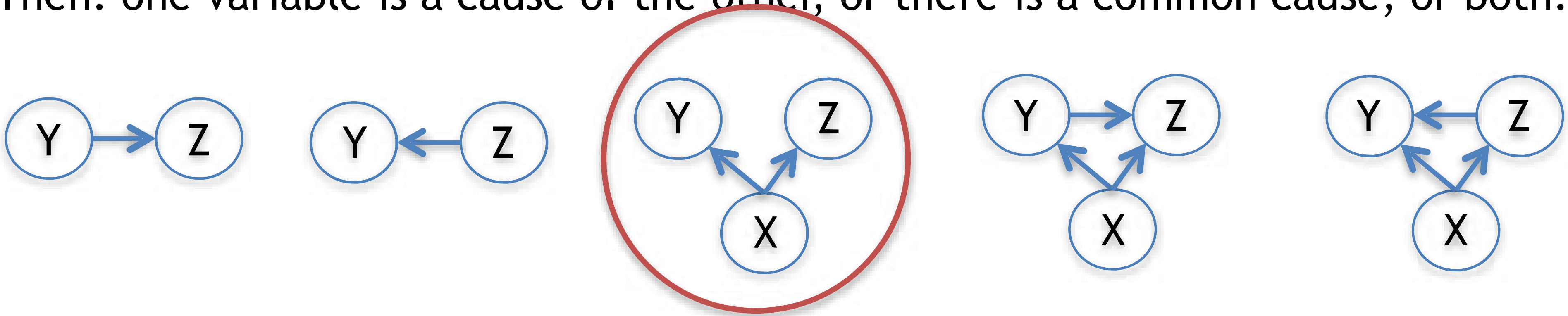

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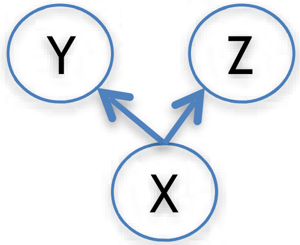
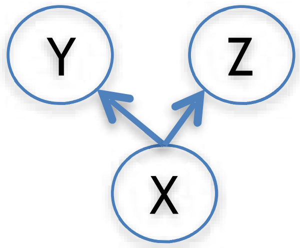


If X is a complete common cause of Y and Z, and Y is not a cause of Z and Z is not a cause of Y, then:

Y and Z are *conditionally independent given X* :

$$P(YZ \mid X) = P(Y \mid X)P(Z \mid X)$$

Reichenbach's principle



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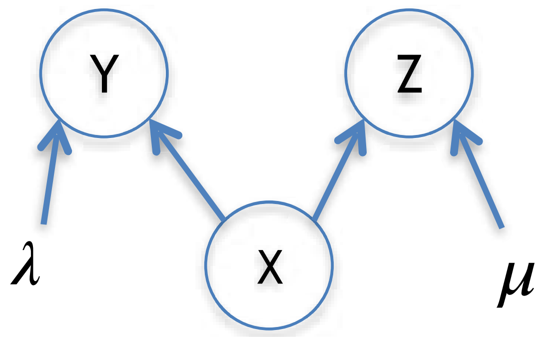
where does this come from ?

Dilation to functions

Suppose that a more complete description is possible in terms of hidden, or latent, variables λ, μ .



Suppose that $Y = f_Y(\lambda,\ X) \quad Z = f_Z(\mu,\ X)$

i.e., Y and Z are functions of earlier variables, and *Y* does not depend on μ and *Z* does not depend on λ .

In this situation we will say that *X* is the complete common cause of *Y* and *Z*.

Suppose further that $P(\lambda X \mu) = P(\lambda)P(X)P(\mu)$.
Then $P(YZ \mid X) = P(Y \mid X)P(Z \mid X)$.

Dilation to functions

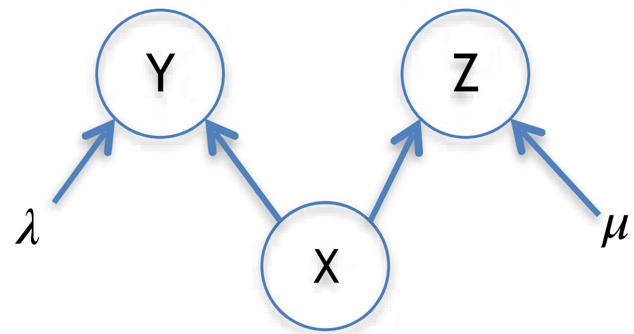
This goes in the other direction too. Hence:

Theorem

Given a conditional distribution, *P(YZ|X)*, the following are equivalent:

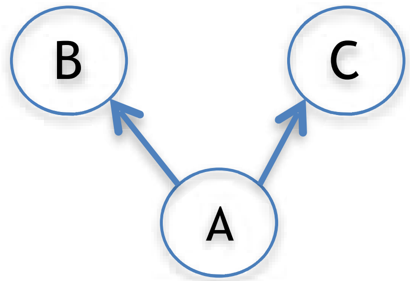
(i) It is possible to define random variables λ, μ, and functions f_Y, f_Z, such that $Y = f_Y(\lambda, X)$, $Z = f_Z(\mu, X)$, and $P(\lambda X \mu) = P(\lambda)P(X)P(\mu)$.

(ii) $P(YZ|X) = P(Y|X)P(Z|X)$



The quantum notion of common cause

Suppose that we take the causal situation to be as shown.



What do the arrows mean?

Following the classical discussion, we expect the arrows to be telling us, that A is (in some sense) a complete common cause of B and C.

Suppose that there is some quantum channel from A to BC.

We should then expect that “A is the complete common cause of B and C” places a constraint on this channel, analogous to the classical factorisation of P(YZ|X).

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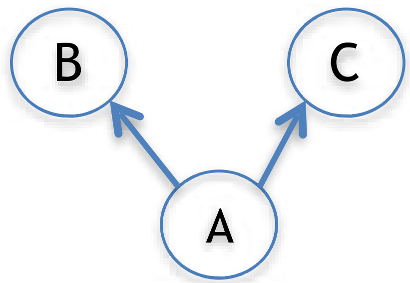

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Idea: Assume an underlying unitary transformation

Notation:

Consider a quantum channel, with input *A* and output *B*, corresponding to a CP map E:

$$\rho_B = E(\rho_A) \, .$$

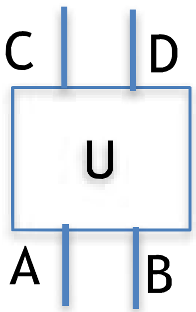
Let the Choi-Jamiołkowski-isomorphic operator be given by:

$$\rho_{B|A} = \sum_{ij} E(|i\rangle_A \langle j|) \otimes |i\rangle_{A^*} \langle j|$$

$\rho_{B|A}$ is a positive operator, with $Tr_B\left(\rho_{B|A}\right) = I_{A^*}$.

Definition:

For a generic bipartite unitary *U*:

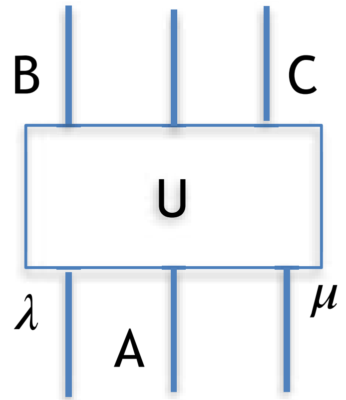


say that *B* *does not influence* *C* if:

- for all inputs ρ_A, the marginal ρ_C is independent of ρ_B
- equivalently, $Tr_D \, \rho_{CD|AB} = \rho_{C|A} \otimes I_B$

Definition:

Given a unitary *U*:



say that *A* is the *complete common cause* of *B* and *C* if:

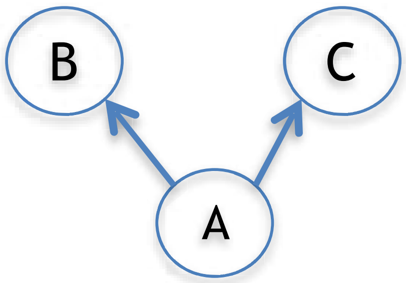
μ does not influence B
λ does not influence C

Theorem (J.-M. Allen, et al., PRX **7**, 031021 (2017)) :

Given $\rho_{BC|A}$, the following are equivalent:

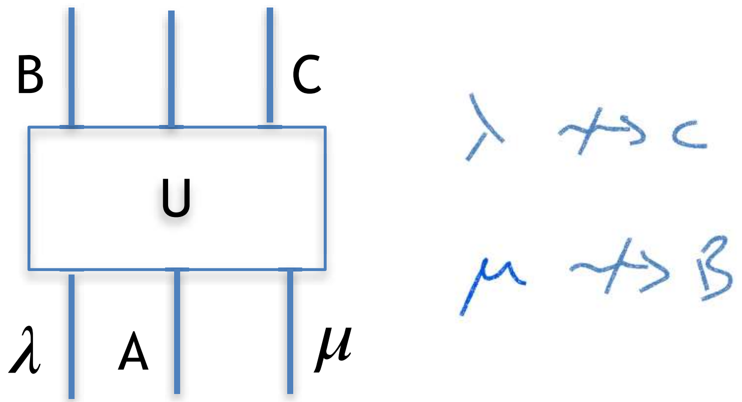
(i) there exists a unitary dilation of $\rho_{BC|A}$, with latent systems λ and μ, such that A is the complete common cause of B and C, and $\rho_{\lambda A \mu} = \rho_\lambda \otimes \rho_A \otimes \rho_\mu$

(ii) $\rho_{BC|A} = \rho_{B|A}\rho_{C|A}$



$\rho_{BC|A} = \rho_{B|A}\rho_{C|A}$

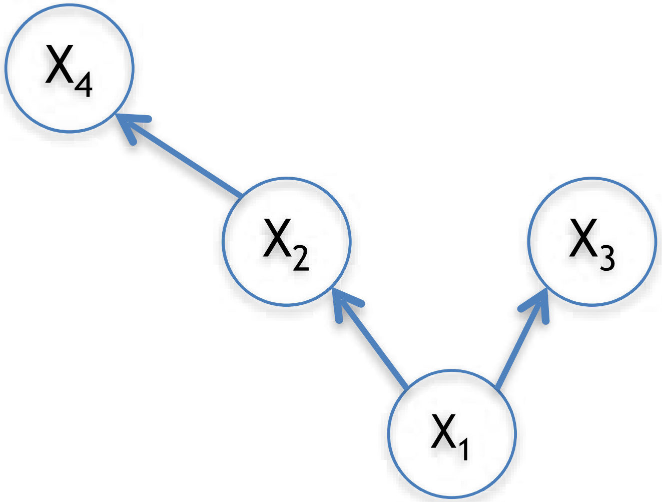
$\Leftrightarrow$



Causal models

Classical causal models

A *directed acyclic graph* with random variables on nodes:



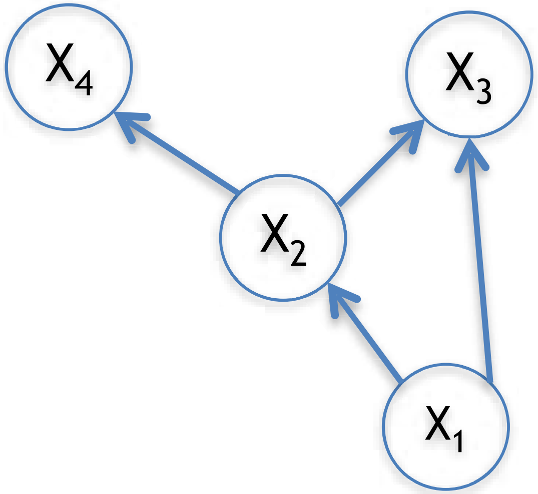
A set of conditional probabilities: For each *i*, $P(X_i \mid Pa_i)$

Joint distribution:
$P(X_1 \ldots X_k) = \Pi_i P(X_i \mid Pa_i)$

Pa_i denotes the parents of X_i, that is the set of nodes X_k such that there is an arrow from X_k to X_i.

Quantum causal models

A *directed acyclic graph*.
Each node is associated with a Hilbert space $H_i \otimes H_i^*$



- A set of channels:

$$\rho_{X_i|\, Pa_i} \in B(H_i \otimes_{k \in Pa_i} \mathrm{H}_\mathrm{k}^*)$$

such that for all i,j $[\, \rho_{X_i|\, Pa_i} \,, \; \rho_{X_j|\, Pa_j}] = 0.$

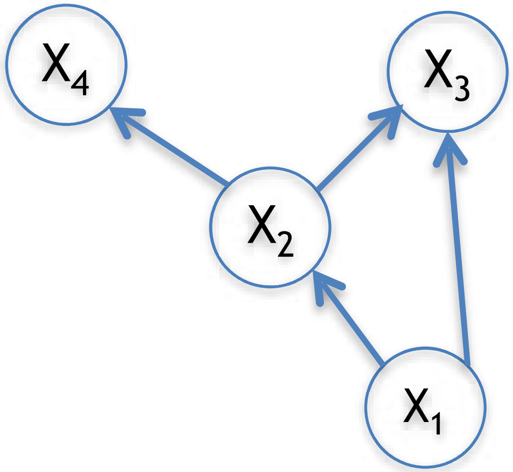
- Form a process operator by taking the product of these channel operators.

- E.g., for the graph on the left:

$$\sigma = \rho_{X_4|\, X_2} \rho_{X_3|\, X_2 X_1} \rho_{X_2|\, X_1} \rho_{X_1}$$

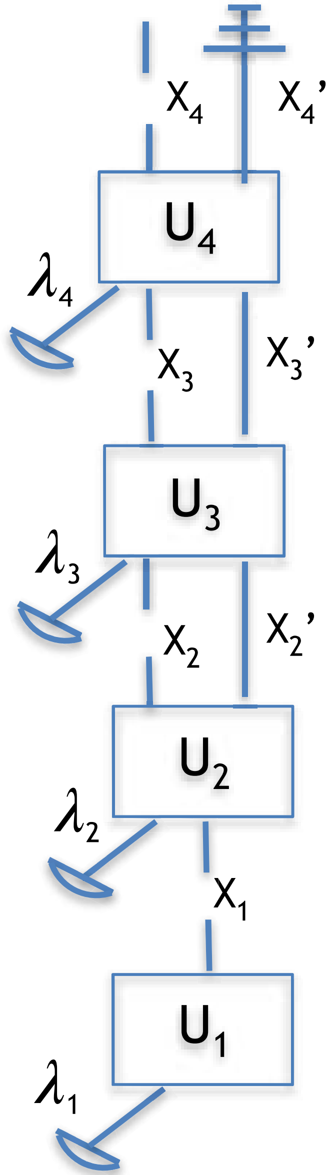
- In general, $\sigma \in B\,(\otimes_i (H_i \otimes H_i^*))$.

See JB, R. Lorenz, O. Oreshkov, arXIv:1906.10726 for the full justification of the quantum definition.



$$\sigma = \rho_{X_4| X_2} \rho_{X_3| X_2 X_1} \rho_{X_2| X_1} \rho_{X_1}$$

$$\Leftrightarrow$$



Independence and causal structure

In classical causal models, the structure of the DAG places constraints on the joint probability distribution in the form of conditional independences.

Definition:

Consider a DAG G. Let S, T and U be disjoint subsets of nodes of G. A path from S to T is an undirected path in the DAG, which starts on an S node and ends on a T node. A path from S to T is blocked by U if any of the following hold:

(i) The path contains a fork at a node in U.
(ii) The path contains a traversal at a node in U.
(iii) The path contains a collider at a node which is not in U, and which does not have any descendants in U.

Say that *S and T are d-separated by U* if all paths from S to T are blocked by U.

Theorem (see, e.g., Pearl, *Causality*):

d-separation is sound and complete for $P(ST|U) = P(S|U)\, P(T|U)$.

Independence in quantum process operators

Reminder: Given three random variables, X,Y,Z, the following are equivalent ways of defining "Y and Z are conditionally independent given X":

P(YZ|X) = P(Y|X) P(Z|X)
I(Y:Z|X) = 0
P(XYZ) P(X) = P(YX) P(ZX)
P(XYZ) = $\alpha(YX)\beta(ZX)$, for real valued functions α and β.

Definition: Given a quantum process operator σ_{STU} say that ***S and T are strongly independent relative to U*** **if** $\sigma_{STU} = \alpha_{SU}\,\beta_{TU},$ for Hermitian operators α and β.

Implied operational statement:

There exists a (global) intervention at the U nodes, such that for any interventions at S, T, the outcomes at S and T are conditionally independent given the outcome at U.

Theorem (JB, R. Lorenz, O. Oreshkov, arXIv:1906.10726):

Consider a quantum causal model, with DAG G, and process operator σ

Suppose that S, T, U are disjoint subsets of nodes, and let R be all nodes not in S,T,U.

Then:

S and T are d-separated by U	$\Rightarrow$	For all local interventions at the R nodes, S and T are strongly independent relative to U

Conclusions

- Following a close look at the notions of common cause and independence, we have given a definition for quantum causal models, and the quantum version of a central result from the classical framework: that d-separation implies conditional independence.

- See arXIv:1906.10726 for many other results, including full justification of the quantum definition, classical split node models and a quantum generalisation of Pearl’s do-calculus.

More foundational and/or speculative remarks

- The classical formalism has a natural interpretation, wherein causal structure is explained in terms of underlying functional relationships between variables. The functional relationships are taken to be facts about the world (*ontic*). Probabilities arise when an agent does not know the values of all variables, hence expresses degrees of belief with a probability distribution (*epistemic*).

- The existence of a compelling quantum analogue to the classical formalism, with unitaries replacing functions, lends support to the view that the unitaries (and the causal structure they define) are ontic, and that the positive operators are epistemic.

- But the positive operators express … information about what?

- The quantum formalism uses split nodes (two Hilbert spaces per node). A split node classical formalism can also be written down, more closely analogous to the quantum case than the traditional classical formalism. Is there a successful quantum formalism involving only a single Hilbert space per node? If not, why not?