Bounding correlations in networks

Antonio Acín
ICREA Professor at ICFO-Institut de Ciencies Fotoniques, Barcelona
AXA Chair in Quantum Information Science

Causality in the quantum world: harnessing quantum effects in causal inference problems, Anacapri, Italy, 19 September 2019

Physical correlations

Physical principles translate into limits on correlations.

Physical correlations

Physical principles translate into limits on correlations.

No-signalling correlations: correlations compatible with the no-signalling principle, i.e. the impossibility of instantaneous communication.

$$\sum_{a_{k+1},\ldots,a_N} p(a_1,\ldots,a_N|x_1,\ldots,x_N) = p(a_1,\ldots,a_k|x_1,\ldots,x_k)$$

Physical correlations

Physical principles translate into limits on correlations.

No-signalling correlations: correlations compatible with the no-signalling principle, i.e. the impossibility of instantaneous communication.

$$\sum_{a_{k+1},\ldots,a_N} p(a_1,\ldots,a_N|x_1,\ldots,x_N) = p(a_1,\ldots,a_k|x_1,\ldots,x_k)$$

$p(a_1|x_1)$

Physical correlations

Classical correlations: correlations established by classical means.

$$p\left(a_1,\ldots,a_N \middle| x_1,\ldots,x_N\right) = \sum_\lambda p(\lambda) D\left(a_1 \middle| x_1,\lambda\right)\ldots D\left(a_N \middle| x_N,\lambda\right)$$

These are the standard “EPR” correlations. Independently of fundamental issues, these are the correlations achievable by classical resources. Bell inequalities define the limits on these correlations.

Physical correlations

Quantum correlations: correlations established by quantum means.

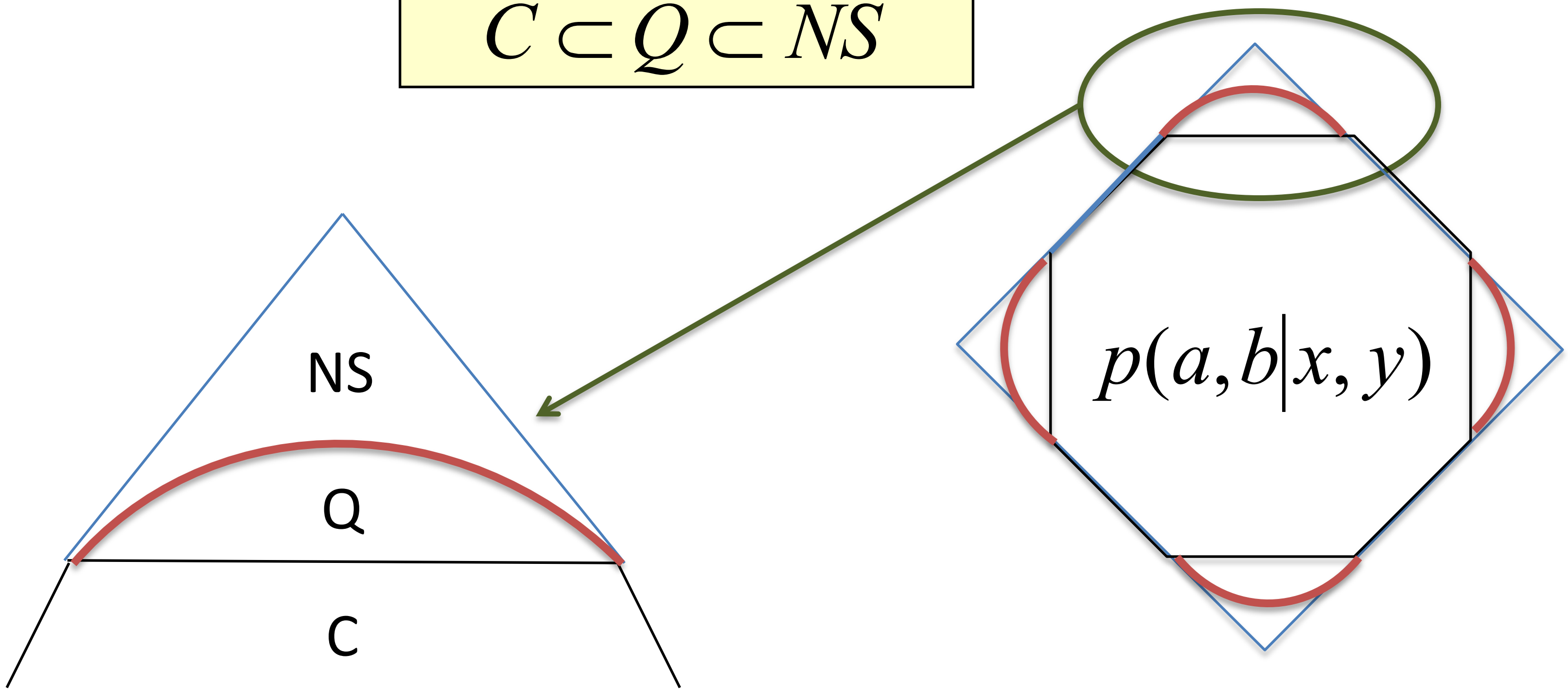
$$p(a_1,\ldots,a_N | x_1,\ldots,x_N) = \langle \Psi | M_{a_1}^{x_1} \otimes \cdots \otimes M_{a_N}^{x_N} | \Psi \rangle$$

$$\sum_{a_i} M_{a_i}^{x_i} = 1 \qquad M_{a'_i}^{x_i} M_{a_i}^{x_i} = \delta_{a_i a'_i} M_{a_i}^{x_i}$$

Everything is expressed in terms of operators (the quantum state and the measurement projectors) acting on a Hilbert space.

Physical correlations

$$C \subset Q \subset NS$$



Characterization of Quantum Correlations

Navascués, Pironio, Acin, PRL 2007, NJP 2009

Characterizing quantum correlations

Given $p(a,b|x,y)$, does it have a quantum realization?

$$p\left(a,b\middle|x,y\right)=\left\langle\Psi\middle|M_a^x\otimes M_b^y\middle|\Psi\right\rangle$$

$$\sum_a M_a^x = 1$$

$$M_a^x M_{a'}^x = \delta_{a'a} M_a^x$$

Example:

$$p\left(a,b\middle|0,0\right)=p\left(a,b\middle|0,1\right)=p\left(a,b\middle|1,0\right)=\frac{1}{8}\left(2+\sqrt{3},2-\sqrt{3},2-\sqrt{3},2+\sqrt{3}\right)$$

$$p\left(a,b\middle|1,1\right)=\left(0.245,0.255,0.255,0.245\right)$$

Previous work by Tsirelson

NPA in a nutshell

$$p(ab|xy) \overset{?}{=} \langle\Psi|\Pi_{a|x}\Pi_{b|y}|\Psi\rangle \qquad [\Pi_{a|x}, \Pi_{b|y}] = 0$$

NPA in a nutshell

$$p(ab|xy) \overset{?}{=} \langle\Psi|\Pi_{a|x}\Pi_{b|y}|\Psi\rangle \qquad [\Pi_{a|x}, \Pi_{b|y}] = 0$$

Idea: **assume** you had the state Ψ and measurements $\Pi_{a|x}$and $\Pi_{b|y}$ producing the correlations.

Then, for any set of operators made of products of the measurements operators, X, the matrix γ with elements $\gamma_{ij} = \langle\Psi|X_i^\dagger X_j|\Psi\rangle$ is positive semi-definite.

NPA in a nutshell

$$p(ab|xy) \stackrel{?}{=} \langle\Psi|\Pi_{a|x}\Pi_{b|y}|\Psi\rangle \qquad [\Pi_{a|x}, \Pi_{b|y}] = 0$$

Idea: **assume** you had the state Ψ and measurements $\Pi_{a|x}$ and $\Pi_{b|y}$ producing the correlations.

Then, for any set of operators made of products of the measurements operators, X, the matrix γ with elements $\gamma_{ij} = \langle\Psi|X_i^\dagger X_j|\Psi\rangle$ is positive semi-definite.

Example: $X = \{\Pi_{a|x}\} \cup \{\Pi_{b|y}\}$

$$\gamma = \begin{array}{cc} \begin{array}{cc} \Pi_{a|x} & \Pi_{b|y} \end{array} & \\ \left[\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right] & \begin{array}{c} \Pi_{a|x} \\ \Pi_{b|y} \end{array} \end{array}$$

NPA in a nutshell

$$p(ab|xy) \overset{?}{=} \langle\Psi|\Pi_{a|x}\Pi_{b|y}|\Psi\rangle \qquad \left[\Pi_{a|x}, \Pi_{b|y}\right] = 0$$

Idea: **assume** you had the state Ψ and measurements $\Pi_{a|x}$ and $\Pi_{b|y}$ producing the correlations.

Then, for any set of operators made of products of the measurements operators, X, the matrix γ with elements $\gamma_{ij} = \langle\Psi|X_i^\dagger X_j|\Psi\rangle$ is positive semi-definite.

Example: $X = \{\Pi_{a|x}\} \cup \{\Pi_{b|y}\}$

$$\gamma = \begin{array}{cc} \begin{array}{cc} \Pi_{a|x} & \qquad\qquad \Pi_{b|y} \end{array} & \\ \left[\begin{array}{c:c} \qquad\qquad & \begin{matrix}\langle\Psi|\Pi_{a|x}\Pi_{b|y}|\Psi\rangle \\ || \\ p(ab|xy)\end{matrix} \\ \hdashline & \\ & \end{array}\right] & \begin{array}{c} \Pi_{a|x} \\ \\ \\ \Pi_{b|y}\end{array} \end{array}$$

NPA in a nutshell

$$p(ab|xy) \overset{?}{=} \langle\Psi|\Pi_{a|x}\Pi_{b|y}|\Psi\rangle \qquad \left[\Pi_{a|x},\Pi_{b|y}\right] = 0$$

Idea: **assume** you had the state Ψ and measurements $\Pi_{a|x}$and $\Pi_{b|y}$ producing the correlations.

Then, for any set of operators made of products of the measurements operators, X, the matrix γ with elements $\gamma_{ij} = \langle\Psi|X_i^\dagger X_j|\Psi\rangle$ is positive semi-definite.

Example: $X = \{\Pi_{a|x}\} \cup \{\Pi_{b|y}\}$

$$\langle\Psi\big|\Pi_{a|x}\Pi_{a'|x}\big|\Psi\rangle = p_A(a|x)\delta_{aa'}$$

$$\gamma = \begin{array}{cc} \begin{array}{cc} \Pi_{a|x} & \Pi_{b|y} \end{array} & \\ \left[\begin{array}{c|c} & \begin{matrix}\langle\Psi|\Pi_{a|x}\Pi_{b|y}|\Psi\rangle \\ || \\ p(ab|xy)\end{matrix} \\ \hline & \end{array}\right] & \begin{array}{c} \Pi_{a|x} \\ \\ \Pi_{b|y} \end{array} \end{array}$$

$$\langle\Psi\big|\Pi_{a|x}\Pi_{a'|x'}\big|\Psi\rangle = \gamma_{a|x,a'|x'}$$

unknown variables

NPA in a nutshell

Can one find values for the unknown terms involving non-commuting local measurements, e.g. $\gamma_{a|x,a'|x'}$, such that $\gamma \geq 0$? If not, the correlations are not quantum.

This can be answered through SDP.

Step n in the hierarchy is defined by the set of products of n measurement operators.

NPA in a nutshell

Can one find values for the unknown terms involving non-commuting local measurements, e.g. $\gamma_{a|x,a'|x'}$, such that $\gamma \geq 0$? If not, the correlations are not quantum.

This can be answered through SDP.

Step n in the hierarchy is defined by the set of products of n measurement operators.

Example: 1+AB

$$\gamma_{1+AB} = \begin{array}{c|c|c|c} & \Pi_{a|x} \quad \Pi_{b|y} & \Pi_{a|x}\Pi_{b|y} & \\ \hline & \left.\begin{array}{c} \gamma_1 \end{array}\right. & & \begin{array}{c} \Pi_{a|x} \\ \Pi_{b|y} \end{array} \\ \hline & & & \Pi_{a|x}\Pi_{b|y} \end{array}$$

NPA in a nutshell

Can one find values for the unknown terms involving non-commuting local measurements, e.g. $\gamma_{a|x,a'|x'}$, such that $\gamma \geq 0$? If not, the correlations are not quantum.

This can be answered through SDP.

Step n in the hierarchy is defined by the set of products of n measurement operators.

Example: 1+AB

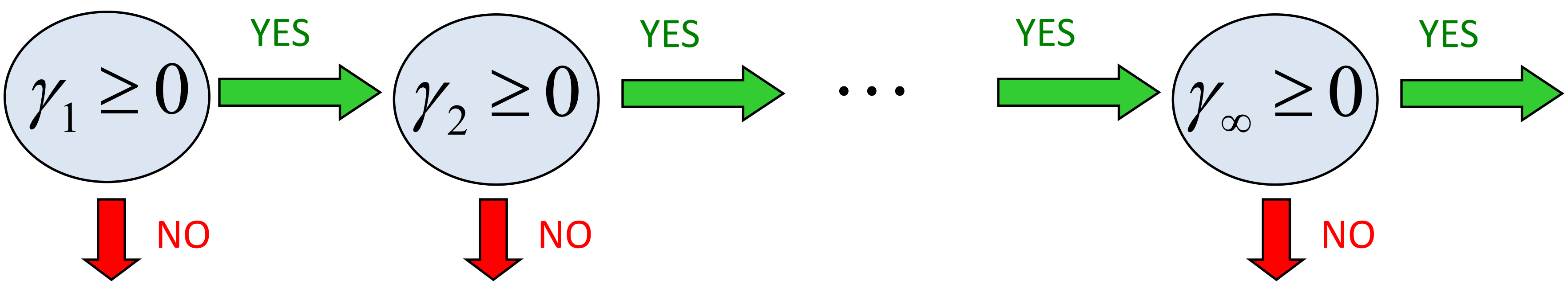
$$\begin{array}{r|cc|c|l} & \Pi_{a|x} & \Pi_{b|y} & \Pi_{a|x}\Pi_{b|y} & \\ \hline & \gamma_1 & & & \Pi_{a|x} \\ \gamma_{1+AB} = & & & & \Pi_{b|y} \\ \hline & & & & \Pi_{a|x}\Pi_{b|y} \\ \hline \end{array}$$

$$\langle\Psi\big|\Pi_{a'|x'}\Pi_{a|x}\Pi_{b|y}\big|\Psi\rangle = 0$$

$$p(ab|xy) = \langle\Psi|\Pi_{a|x}^2\Pi_{b|y}^2|\Psi\rangle$$

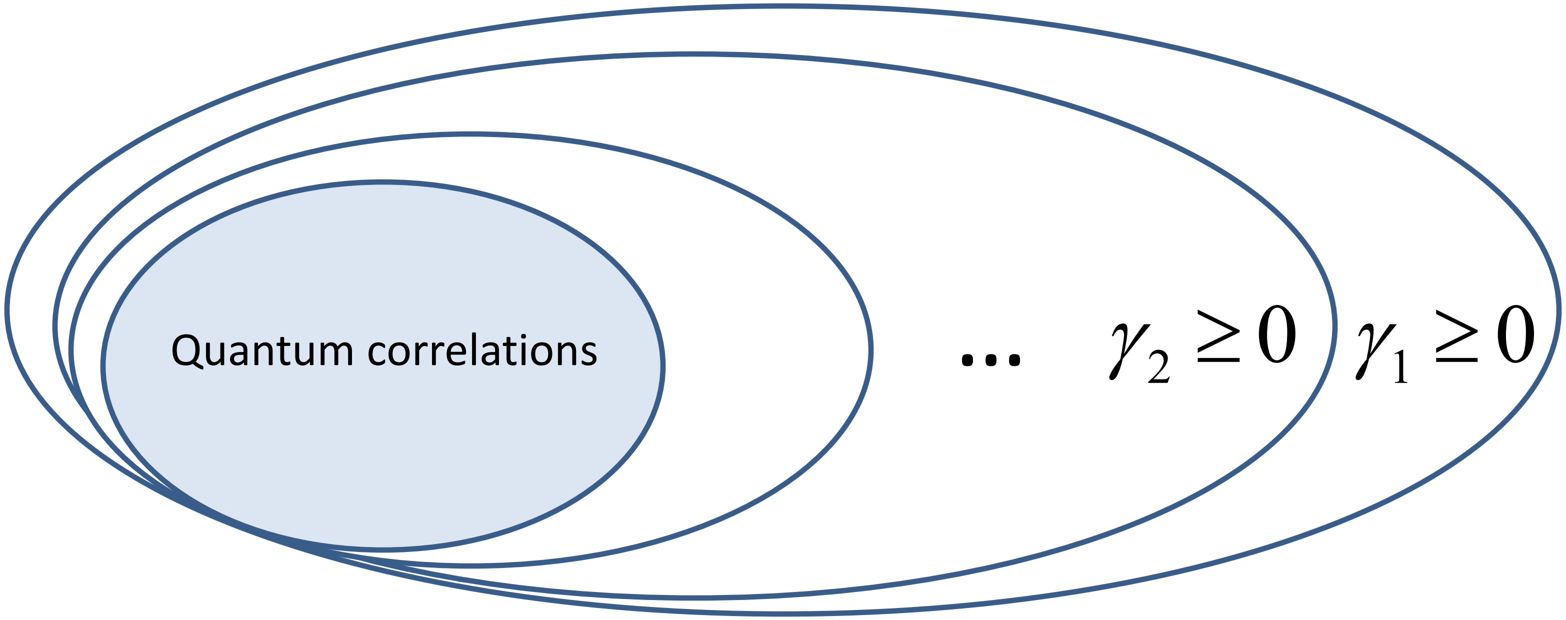
NPA hierarchy

Given a probability distribution $p(a,b|x,y)$, we have defined a hierarchy consisting of a series of tests based on semi-definite programming techniques allowing the detection of supra-quantum correlations.



The hierarchy is asymptotically convergent.

NPA hierarchy



Every step in the hierarchy defines a convex set that is included in the previous step. Convergence is provably attained asymptotically.

In many situations convergence is attained after a few steps. But there is evidence that there may be situations that require an infinite number of steps.

Characterizing quantum correlations

Example:

$$p(a,b|0,0) = p(a,b|0,1) = p(a,b|1,0) = \frac{1}{8}\left(2+\sqrt{3},2-\sqrt{3},2-\sqrt{3},2+\sqrt{3}\right)$$

$$p(a,b|1,1) = (0.245,0.255,0.255,0.245)$$

Solution: it is not quantum, that is, there exists no quantum state of two particles and local measurements acting on them that produce these correlations.

The experimental observation of these correlations would imply the failure of quantum physics, as Bell violations did for classical physics.

Beyond Bell's scenario

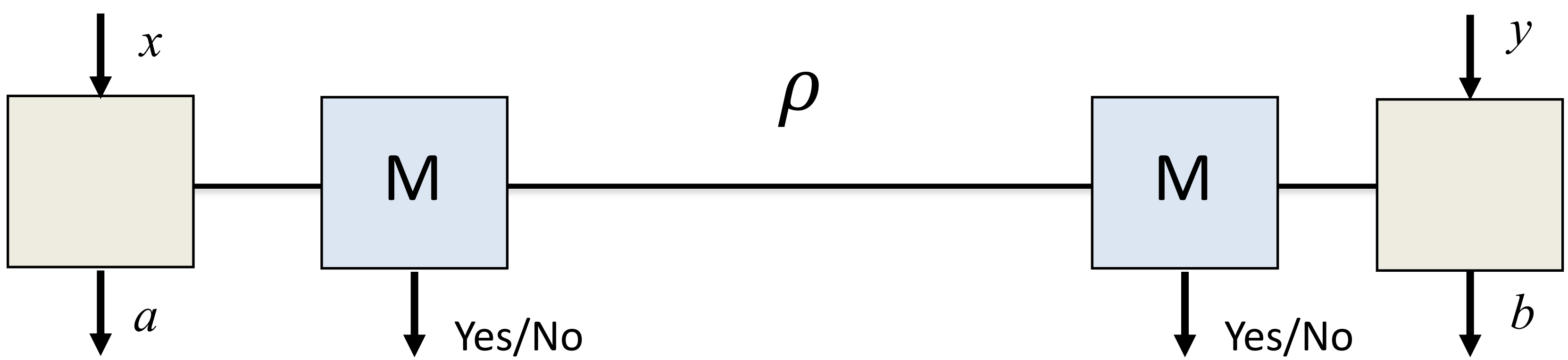
Hidden non-locality

Popescu: there exist entangled states ρ that do not violate any Bell inequality, but that can be mapped by LOCC into a state σ that does it.

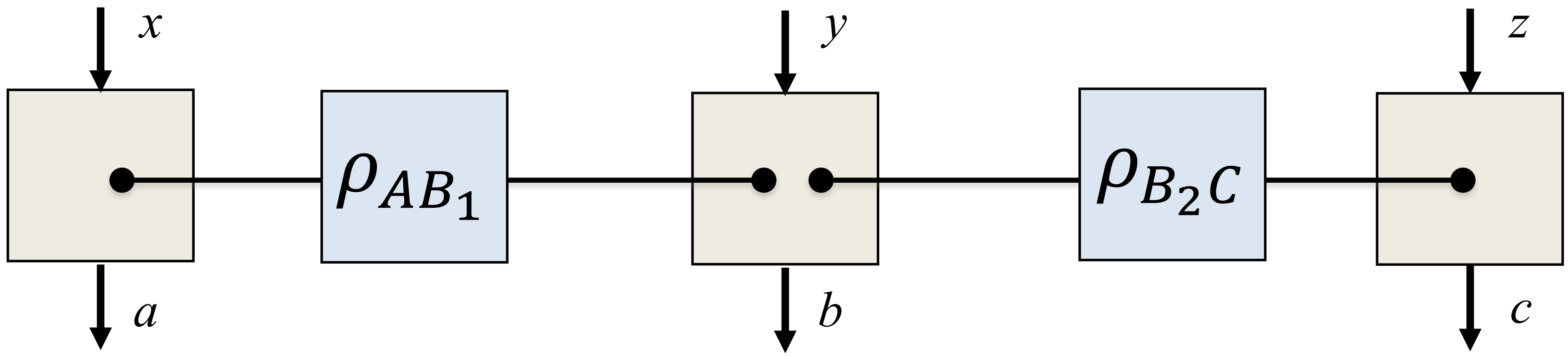
Hidden non-locality

Popescu: there exist entangled states ρ that do not violate any Bell inequality, but that can be mapped by LOCC into a state σ that does it.

Important: the settings for the Bell test should be decided after the LOCC protocol.



Bilocality

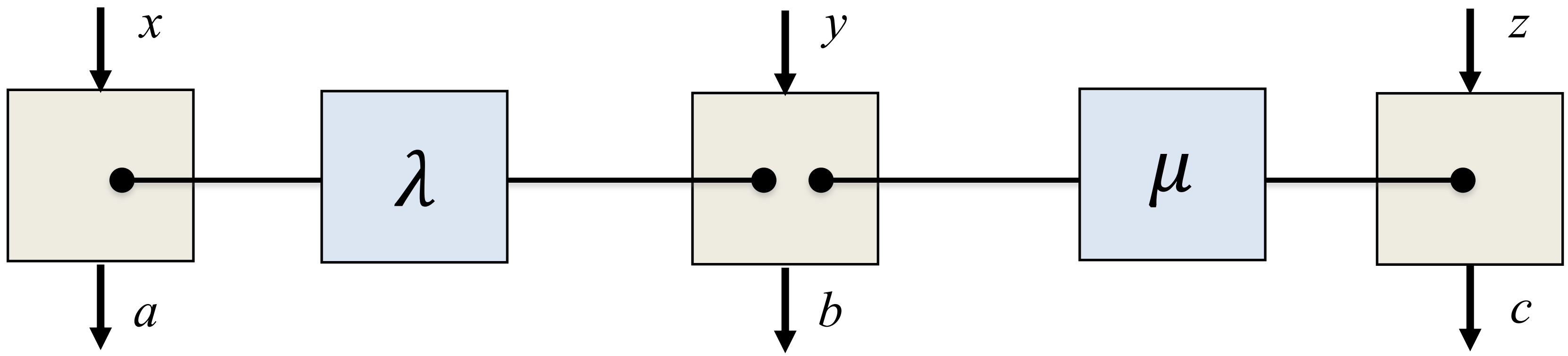


Branciard, Gisin, Pironio, Phys. Rev. Lett.. 2, 184 (2011)

$$p(abc|xyz) = \mathrm{tr}\left(\Pi_{a|x} \otimes \Pi_{b|y} \otimes \Pi_{c|z} \rho_{AB_1} \otimes \rho_{B_2C}\right)$$

$$p(abc|xyz) = \sum_\lambda p(\lambda)\, p(a|\lambda x) p(b|\lambda y)\, p(c|\lambda z)$$

Bilocality



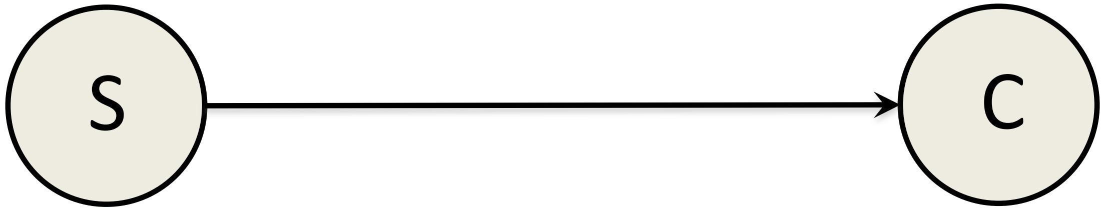
Branciard, Gisin, Pironio, Phys. Rev. Lett.. 2, 184 (2011)

$$p(abc|xyz) = \mathrm{tr}\left(\Pi_{a|x} \otimes \Pi_{b|y} \otimes \Pi_{c|z} \rho_{AB_1} \otimes \rho_{B_2C}\right)$$

$$p(abc|xyz) = \sum_\lambda p(\lambda)\, p(a|\lambda x) p(b|\lambda y)\, p(c|\lambda z)$$

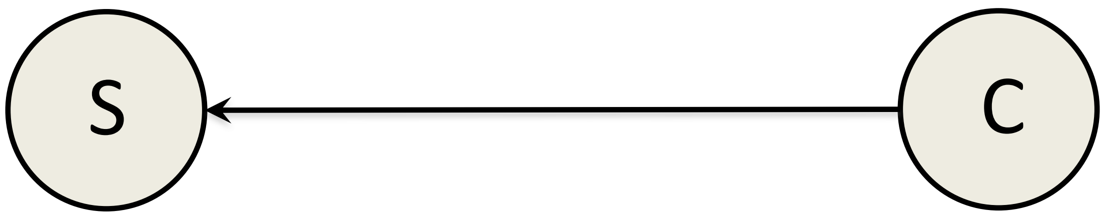
$$p(abc|xyz) = \sum_{\lambda\mu} p(\lambda) p(\mu)\, p(a|\lambda x) p(b|\lambda\mu y)\, p(c|\mu z)$$

Causal networks



Main question: understand the causes that could be behind the observed correlations among a set of random variables.

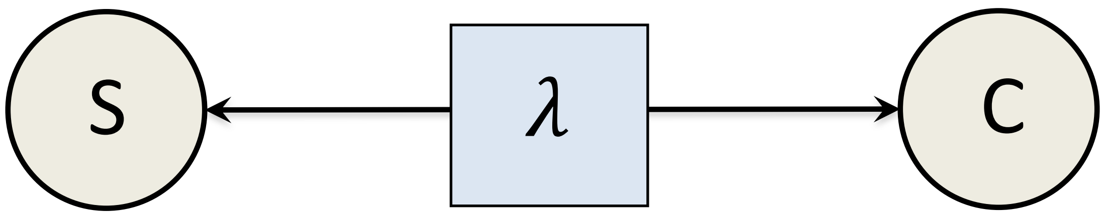
Causal networks



Main question: understand the causes that could be behind the observed correlations among a set of random variables.

Given two correlated variables, either direct causation is possible.

Causal networks



Main question: understand the causes that could be behind the observed correlations among a set of random variables.

Given two correlated variables, either direct causation is possible.

But even more intricate causation patterns could explain the correlations.

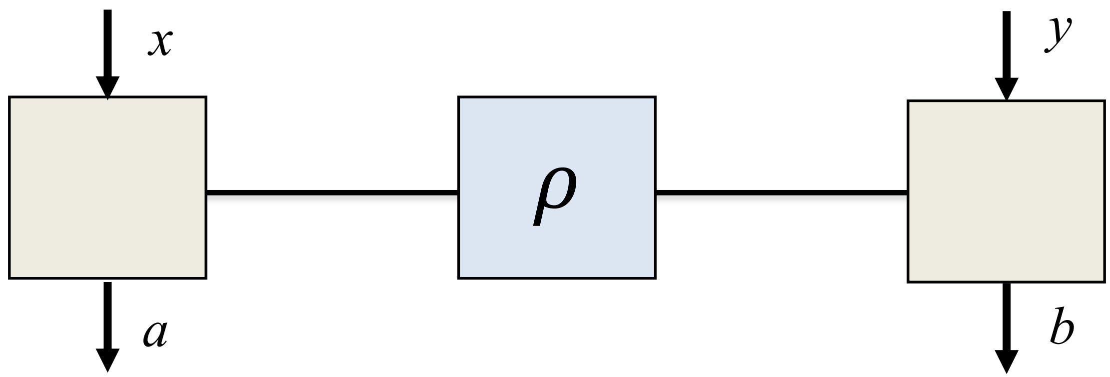
Causal networks

Representation of causality patterns through directed acyiclic graphs. Observed variables are represented by circles, hidden variables by squares and causes by directed edges.

Causal networks

Representation of causality patterns through directed acyiclic graphs. Observed variables are represented by circles, hidden variables by squares and causes by directed edges.

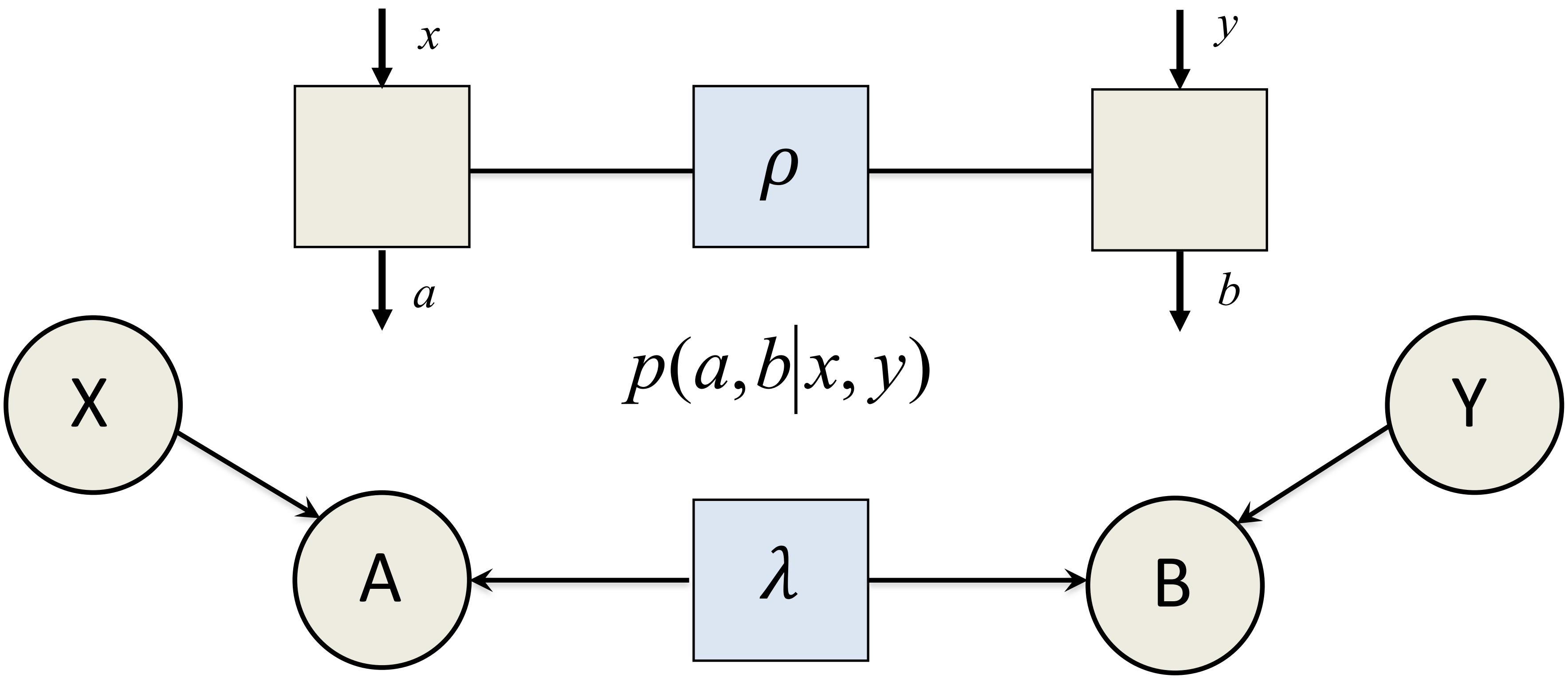
Bell setups can be understood in this language. Fritz, NJP'12; Wood & Spekkens, NJP '15



Causal networks

Representation of causality patterns through directed acyiclic graphs. Observed variables are represented by circles, hidden variables by squares and causes by directed edges.

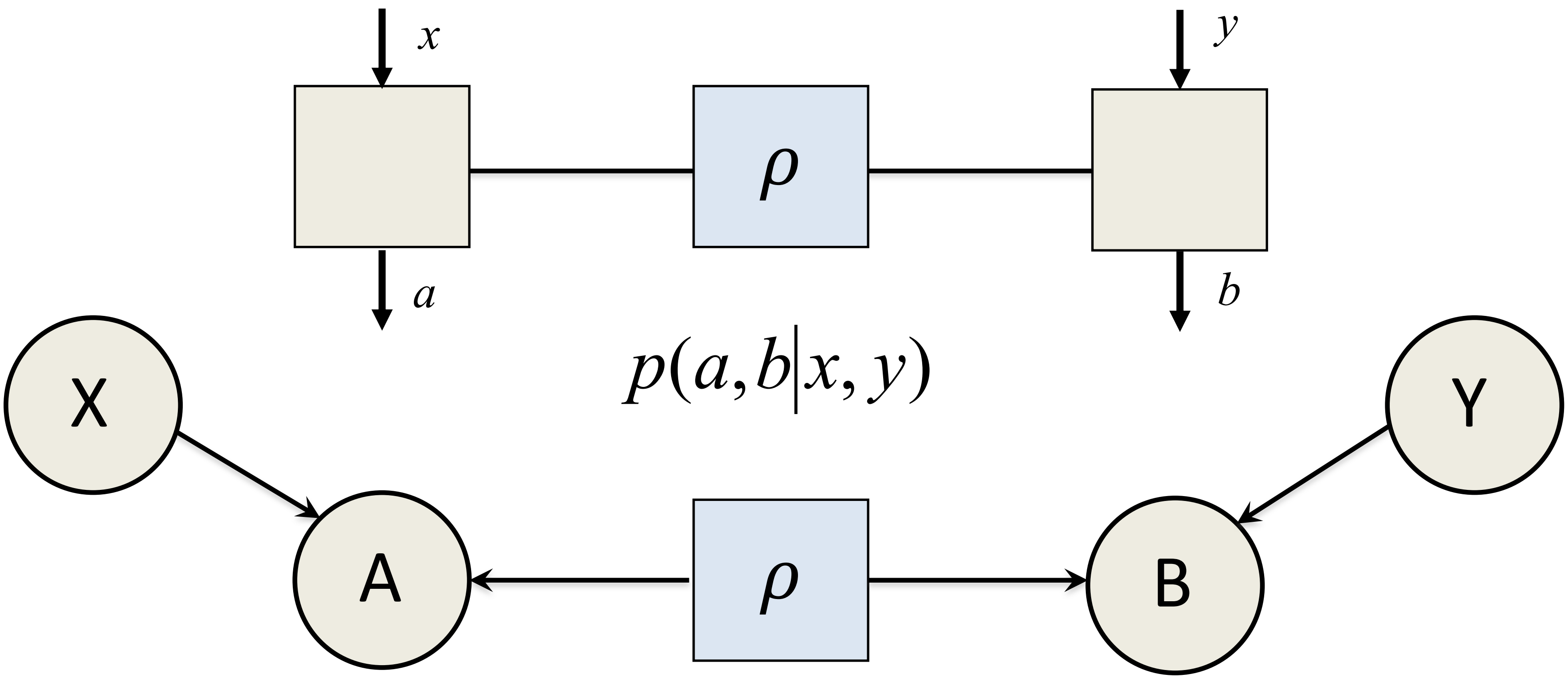
Bell setups can be understood in this language. Fritz, NJP'12; Wood & Spekkens, NJP '15



Causal networks

Representation of causality patterns through directed acyiclic graphs. Observed variables are represented by circles, hidden variables by squares and causes by directed edges.

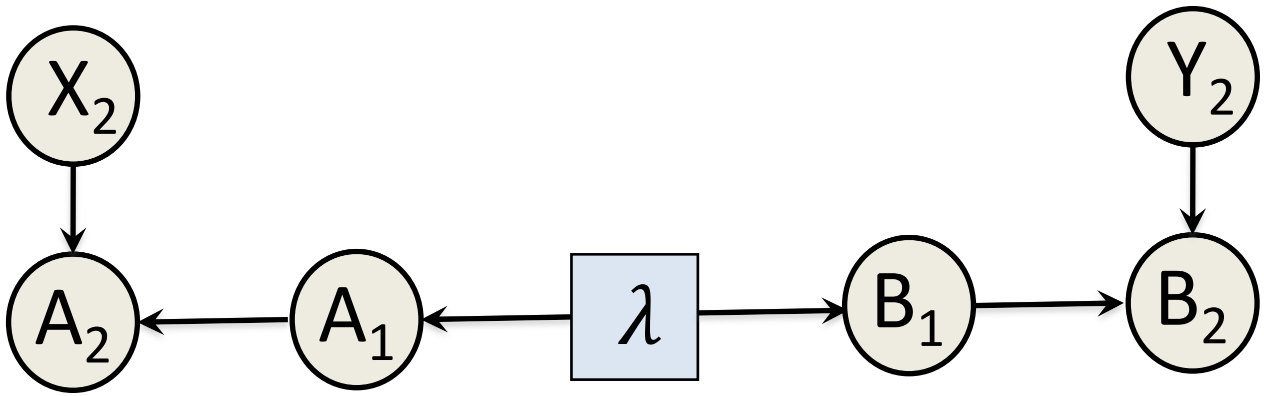
Bell setups can be understood in this language. Fritz, NJP'12; Wood & Spekkens, NJP '15



Causal networks

Generalized Bell scenarios have a natural interpretation in the causal-network scenario.

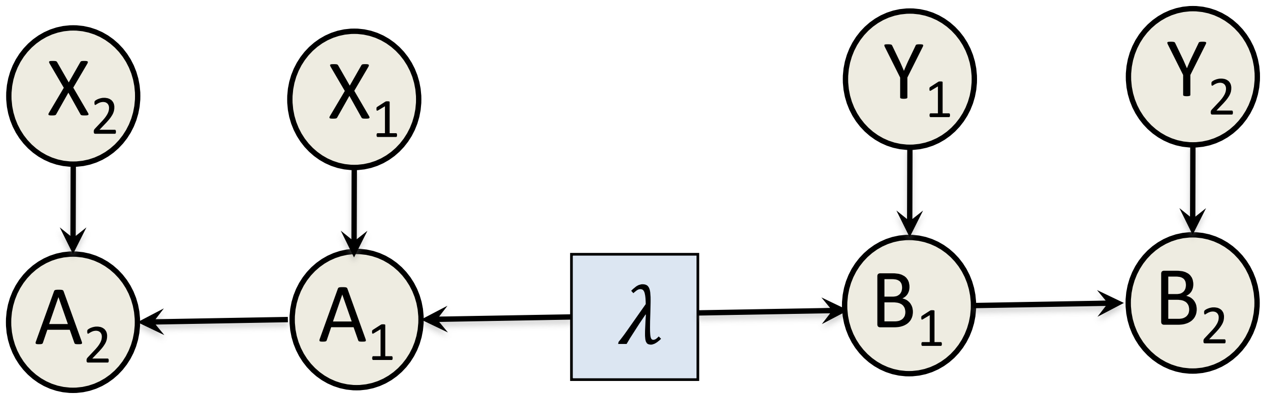
Hidden
non-locality



Causal networks

Generalized Bell scenarios have a natural interpretation in the causal-network scenario.

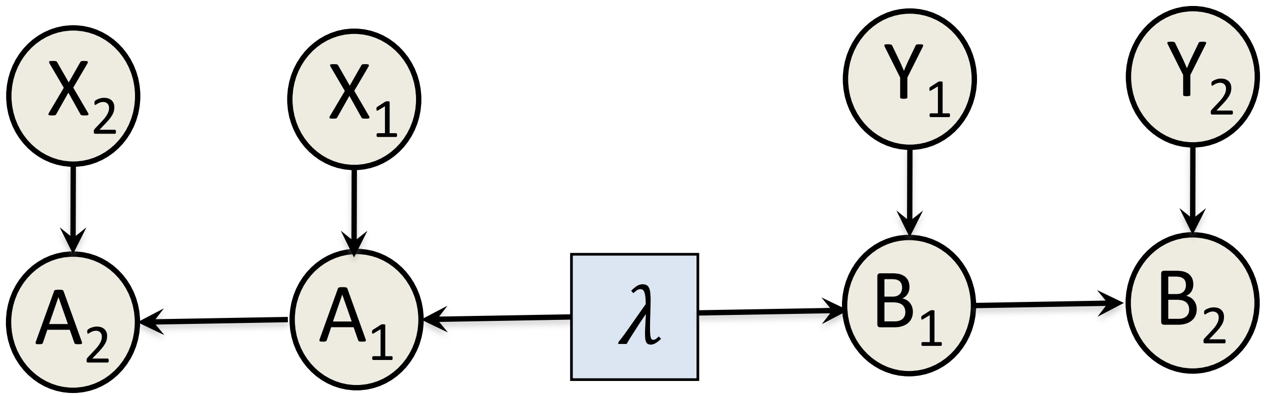
Hidden non-locality



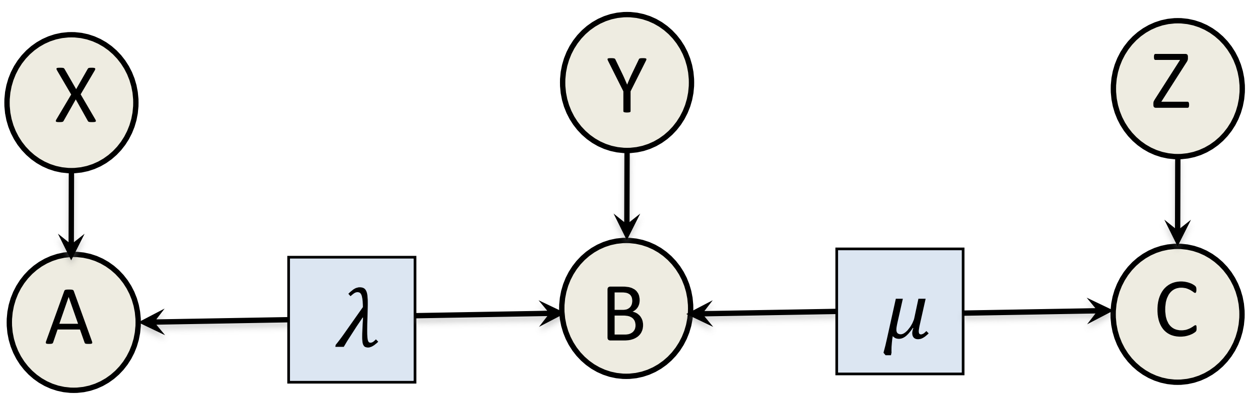
Causal networks

Generalized Bell scenarios have a natural interpretation in the causal-network scenario.

Hidden non-locality



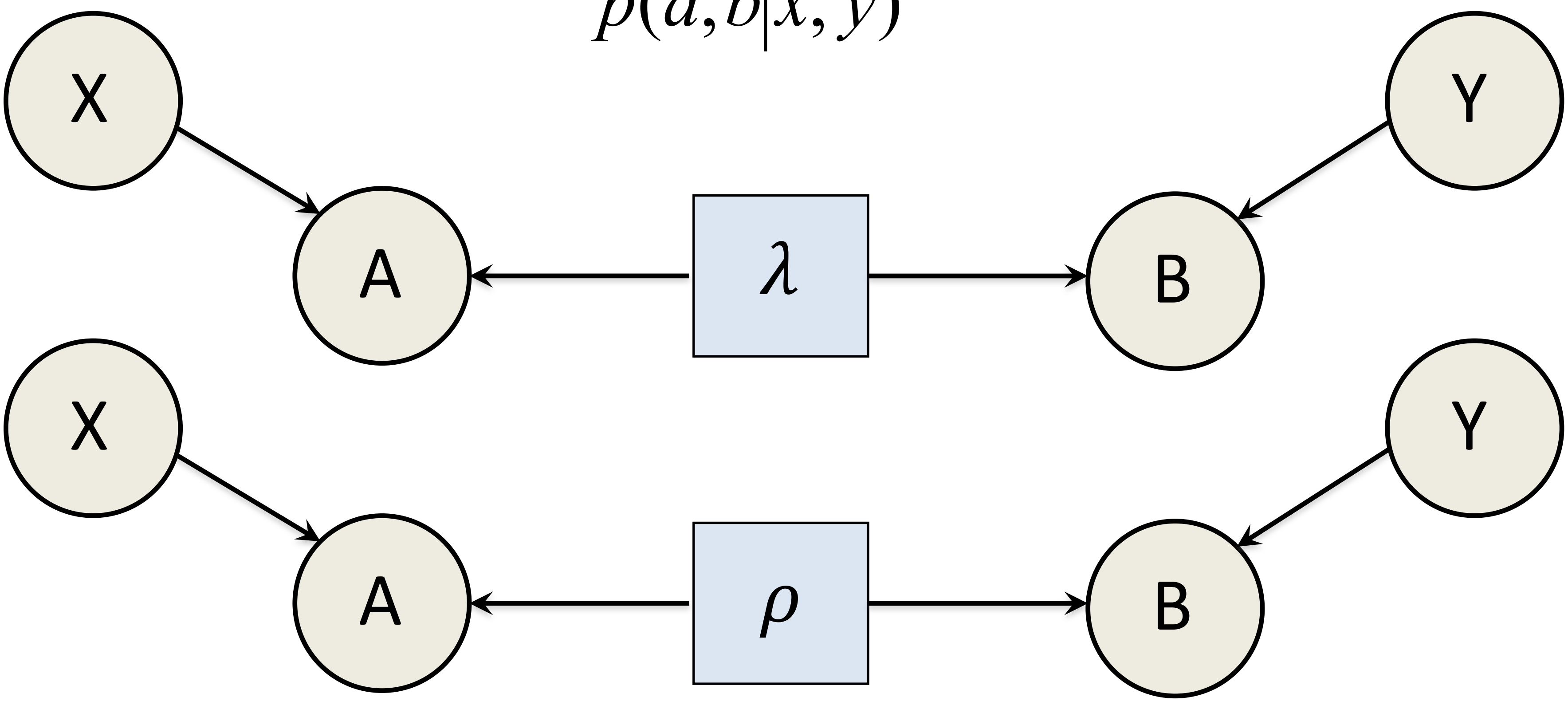
Bilocality



Quantum causality

Bell's theorem: nonlocal correlations can be explained by a quantum causal model, but not by the classical counterpart.

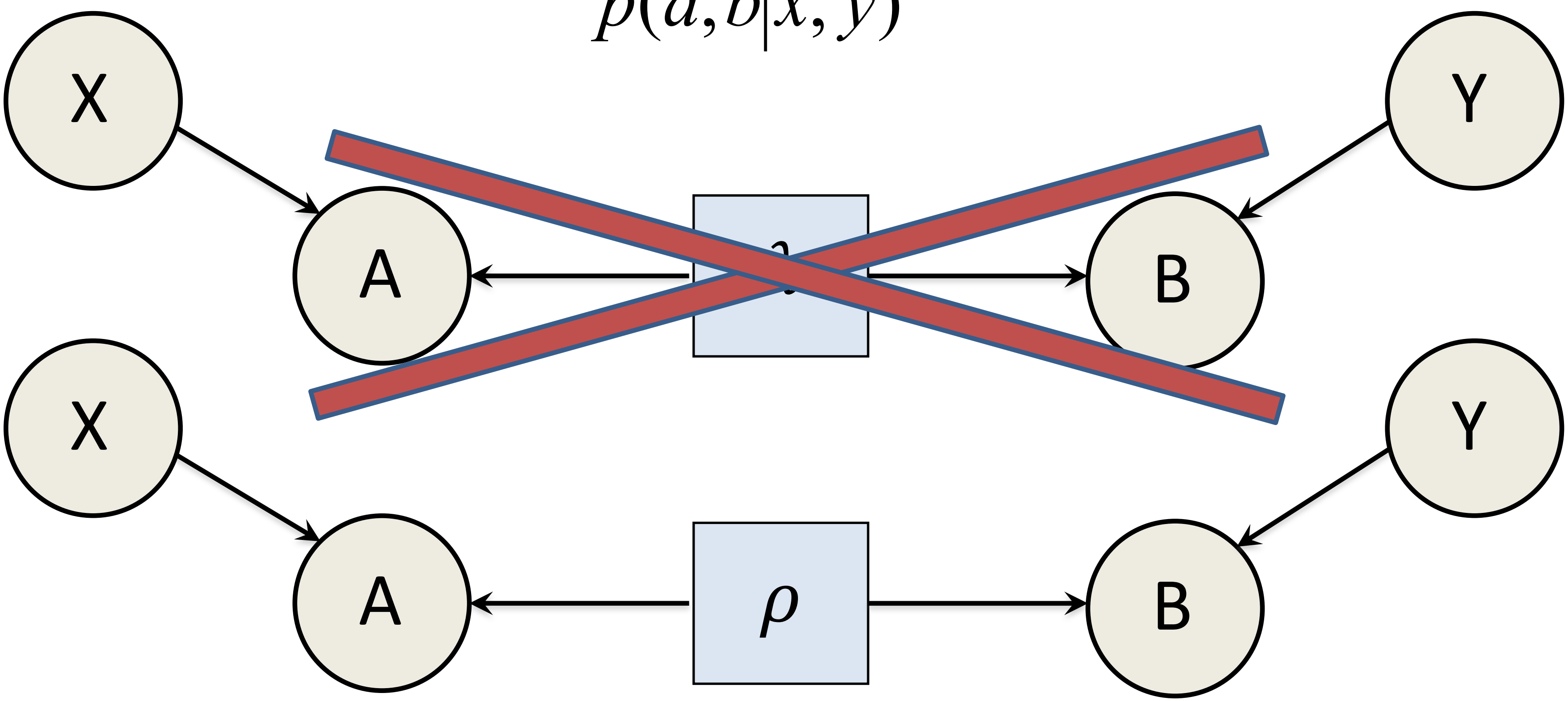
$$p(a,b|x,y)$$



Quantum causality

Bell's theorem: nonlocal correlations can be explained by a quantum causal model, but not by the classical counterpart.

$$p(a,b|x,y)$$

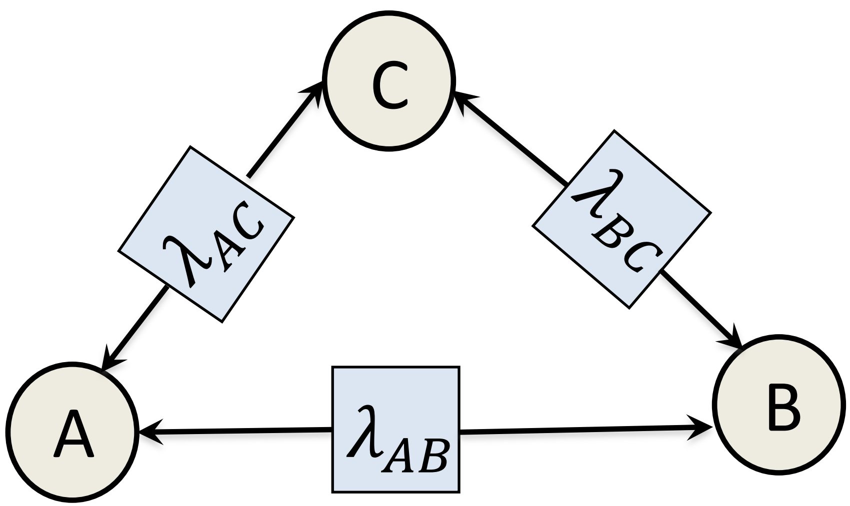


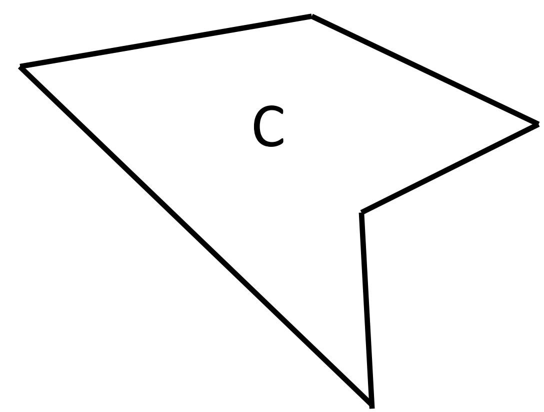
How is causality affected by quantum information?

Causal networks

Make use of this freedom to design stronger “Bell” tests.

Challenge: the sets of correlations are not convex!





Causal networks

Make use of this freedom to design stronger "Bell" tests.

Challenge: the sets of correlations are not convex!